Faculty of Engineering (Shoubra) Benha University Time allowed: 3 hours	2nd year Production Mechanical Department	Engineering Mathematics and Physics Department 16th of January 2011

The Examination consists of 4 questions with 8 sub questions (10 points for each sub question)

ورقة أولى

Question 1

a- Evaluate the following integrals:

i) $\int_0^1 t^{aq-1}(1-t^q)^{b-1}dt$ ii) $\int_0^\infty \frac{e^{2t}-e^{-3t}}{t}dt$ iii) $\int_0^\infty \frac{a\,dt}{\sqrt{t}(a^2+t^2)}$

b- Find Laplace transform for the following functions:

i) $f(t) = t\sin 2t\cosh 3t + e^{-3t}\int_{u=0}^{t} e^{-2u}\sin u\,du$, ii) $g(t) = \begin{Bmatrix} 6 & 0<t\le 2 \\ 3t & 2<t\le 3 \\ 9 & t>3 \end{Bmatrix} + \sin^2 4t\,U(t-2)$

Question 2

a- Find inverse Laplace for the following functions:

i) $F(s) = \frac{s\,e^{-2s}}{s^2+6s+20} + \frac{1}{(s+9)^3}$, ii) $G(s) = \frac{1}{s^2(s^2+4)} + \frac{3}{s^2+6s+5}$

b) Expand in Fourier series the following functions:

$f(x) = \begin{cases} \pi/2 + x, & -\pi \le x \le 0 \\ \pi/2 - x, & 0 < x \le \pi \end{cases}$ $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ -(x-\pi), & \pi < x \le 2\pi \end{cases}$

Question 3

a-i) Solve the integral equation $\int_0^\infty f(x)\sin(\alpha x)\,dx = \begin{cases} 1 & 0\le\alpha<1 \\ 2 & 1\le\alpha<2 \\ 0 & 2\le\alpha \end{cases}$

ii) Solve the following system of differential equations:

$$\frac{dx}{dt} + 5x + 4y = 10,\quad x - \frac{dy}{dt} - y = 0,\ X(0) = 2,\ Y(0) = -4$$

b-i) Find the constants of the following curve $y(x) = \frac{1}{a+bx+cx^2}$ that fit the following data: (-1,2), (3,4), (6,9) using least square method.

ii) Solve using Picard's method up to 2nd approximation the following differential equation: $y'' = x^3(y' + y)$, $y(0) = 1$, $y'(0) = ½$, then find y(0.2) using Euler method, h = 0.1.

ورقة ثانية

Question 4

a-i) Find v(x,y), such that f(z) = u(x,y) + i v(x,y) is analytic, where u(x,y)= e^x(xcosy-ysiny). Express f(z) in terms of z.

ii) Find one of the roots of the interpolating Polynomial satisfy the following data (1,3) , (5,-7) , (-13,4) , (2,47) , (-6,15) using inverse Lagrange interpolation.

b- Evaluate the following integrals:

i) $\int_0^{1/2} t^{m-1} \ln(1/2t)\, dt$,

ii) $\oint_c \frac{\cos(z)}{z^2-6z+5} dz$ where c is the circle |z| = 4.

iii) $\int_2^{\infty} e^{-x^2+4x-4} dx$,

iv) $\oint_c \frac{z^3+5z+7}{(z-i)^2} dz$, c: | z-2| + | z+2| = 6

v) $\oint_c \frac{dz}{(z^2+9)^2}$, using Cauchy's residue theorem c: |z+1-i| = 3.

Good Luck
With all best wishes to succeed
Dr. Khaled El Naggar

Model answer

Answer of Question 1

a- i) Put $x = t^q$**,** therefore $dt = (1/q)\ x^{\frac{1}{q}-1}dx$, thus $\int_0^1 t^{aq-1}(1-t^q)^{b-1}dt = \int_0^1 x^{-1/q}x^a(1-x)^{b-1}\frac{1}{q}x^{(1/q)-1}dx$

$= \frac{1}{q}\int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{1}{q}\beta(a,b)$ (4 marks)

ii) Since $L\{\frac{e^{2t}-e^{-3t}}{t}\} = \int_{s=0}^{\infty}\frac{1}{s-2}-\frac{1}{s+3}ds = Ln\ (s+3) - Ln\ (s-2)$, therefore

$\int_0^{\infty}\frac{e^{2t}-e^{-3t}}{t}dt = Ln\ (s+3) - Ln\ (s-2)\Big|_{s=0} = Ln(3/2)$ (3 marks)

iii) Put $y = t^2$**,** therefore $dt = (1/2)\ y^{-\frac{1}{2}}dy$, therefore $\int_0^{\infty}\frac{a\,dt}{\sqrt{t}(a^2+t^2)} = \frac{a}{2}\int_0^{\infty}\frac{y^{-1/2}dy}{y^{1/4}(a^2+y)} =$

$\frac{a}{2}\int_0^{\infty}\frac{y^{-3/4}dy}{(a^2+y)} = \frac{a}{2}\beta\,(1/4, 3/4) = \frac{a}{\sqrt{2}}\pi$ (3 marks)

b-i) $L\{\ t\sin 2t\} = -\frac{d}{ds}(\frac{2}{S^2+4}) = \frac{4s}{(s^2+4)^2}$, therefore $L\{t\sin 2t\cosh 3t\} = L\{\ t\sin 2t\ (\frac{e^{3t}+e^{-3t}}{2})\}$

$= 2[\frac{(s-3)}{((s-3)^2+4)^2}+\frac{(s+3)}{((s+3)^2+4)^2}]$ (3 marks)

$L\{\ \int_{u=0}^{t} e^{-2u}\sin u\ du\ \} = \frac{1}{s[(s+2)^2+1]}$, therefore $L\{\ e^{-3t}\int_{u=0}^{t} e^{-2u}\sin u\ du\ \} = \frac{1}{(s+3)[(s+5)^2+1]}$

(2 marks)

ii) $L\{\begin{cases} 6 & 0 < t \le 2 \\ 3t & 2 < t \le 3 \\ 9 & t > 3 \end{cases}\} = 6[U(t) - U(t-2)] + 3t[U(t-2) - U(t-3)] + 9\ U(t-3)$

$= 6[U(t) - U(t-2)] + 3(t-2+2)\ U(t-2) - 3(t-3+3)U(t-3) + 9\ U(t-3)$

$$= 6U(t) + 3(t-2)\,U(t-2) - 3(t-3)\,U(t-3) = 6/s + (3/s^2)\,(e^{-2s} - e^{-3s})$$

(3 marks)

$$L\{\sin^2 4t\,U(t-2)\} = L\{(½)(1-\cos 8t)\,U(t-2)\} = L\{(½)(1-\cos 8(t-2+2))\,U(t-2)\}$$

$$= (½)\,L\{U(t-2)\} - (½)\,L\{[\cos 8(t-2)\cos(16) - \sin 8(t-2)\sin(16)]\,U(t-2)\}$$

$$= (½)e^{-2s} - (½)[\cos(16)\frac{s}{(s^2+64)} - \sin(16)\frac{8}{(s^2+64)}]\,e^{-2s}$$

(3 marks)

Answer of Question 2

a-i) Since $F(s) = \dfrac{s\,e^{-2s}}{s^2+6s+20} + \dfrac{1}{(s+9)^3} = \dfrac{(s+3-3)\,e^{-2s}}{(s+3)^2+11} + \dfrac{1}{(s+9)^3}$

$$= [\frac{(s+3)}{(s+3)^2+11} - \frac{3}{(s+3)^2+11}]e^{-2s} + \frac{1}{(s+9)^3}, \text{ therefore}$$

$$f(t) = e^{-3(t-2)}\,[\cos\sqrt{11}\,(t-2) - 3/\sqrt{11}\ \sin\sqrt{11}\,(t-2)]\,U(t-2) + (1/2)\,t^2\,e^{-9t}$$

(5 marks)

ii) $L^{-1}\{\dfrac{1}{s(s^2+4)}\} = (1/2)\displaystyle\int_{u=0}^{t} \sin 2u\,du = (1/4)(1-\cos 2t)$, therefore

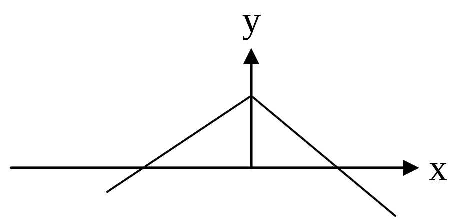
$$L^{-1}\{\frac{1}{s^2(s^2+4)}\} = (1/4)\int_{u=0}^{t} (1-\cos 2u)\,du = (1/4)[t - \sin(2t)/2]$$

(3 marks)

$$L^{-1}\{\frac{3}{s^2+6s+5}\} = L^{-1}\{\frac{3}{(s+3)^2-4}\} = (3/2)\,e^{-3t}\sinh(2t)$$

(2 marks)

b-i) The function is even, therefore



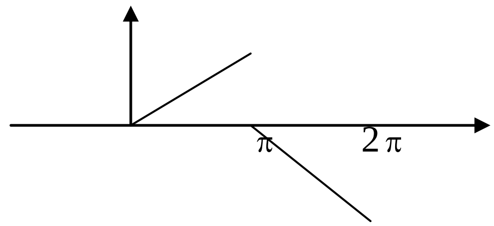
$$a_0 = \frac{2}{\pi}\int_0^{\pi}[(\pi/2) - x]\,dx = 0,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}[(\pi/2) - x]\cos(nx)\,dx = \frac{2}{\pi}[[(\pi/2) - x](\frac{1}{n}\sin(nx)) + (\frac{-1}{n^2}\cos(nx))]_0^{\pi} = \frac{2}{n^2\pi}(1 - \cos n\pi)$$

Therefore $a_{2n} = 0$ and $a_{2n-1} = \dfrac{4}{(2n-1)^2\pi}$, thus $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$

(5 marks)

ii) The function is odd harmonic, therefore



$$a_{2n\text{-}1} = \frac{2}{\pi}\int_0^{\pi} x\cos(2n-1)x\,dx = \frac{2}{\pi}\left(x\left(\frac{\sin(2n-1)x}{2n-1}\right)-\left(\frac{-\cos(2n-1)x}{(2n-1)^2}\right)\right)\Bigg|_0^{\pi} = -\frac{4}{\pi(2n-1)^2}$$

$$b_{2n\text{-}1} = \frac{2}{\pi}\int_0^{\pi} x\,\sin(2n-1)x\,dx = \frac{2}{\pi}\left(x\left(\frac{-\cos(2n-1)x}{2n-1}\right)-\left(\frac{-\sin(2n-1)x}{(2n-1)^2}\right)\right)\Bigg|_0^{\pi} = \frac{2}{(2n-1)}$$

Thus $f(x) = \sum_{n=1}^{\infty} a_{2n\text{-}1}\cos(2n\text{-}1)x + \sum_{n=1}^{\infty} b_{2n\text{-}1}\sin(2n\text{-}1)x$ (5 marks)

Answer of Question 3

a- i) Since $F_S = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\alpha x)\,dx = \sqrt{\frac{2}{\pi}}\begin{cases}1 & 0\le\alpha<1\\ 2 & 1\le\alpha<2\\ 0 & 2\le\alpha\end{cases}$, and Fourier integral is expressed by:

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_S\sin(\alpha x)\,d\alpha = \sqrt{\frac{2}{\pi}}\left[\int_0^{1}\sqrt{\frac{2}{\pi}}\sin(\alpha x)\,d\alpha + \int_1^{2} 2\sqrt{\frac{2}{\pi}}\sin(\alpha x)\,d\alpha\right] = \frac{2}{\pi x}[1+\cos x - 2\cos 2x]$$

(5 marks)

ii) Take Laplace to the above equations such that:

s X(s) - x(0) + 5X(s) + 3 Y(s) = 10/s, X(s) – [s Y(s) - y(0)]- Y(s) = 0, thus

(s +5)X(s) + 3 Y(s) = 10/s + 2 (1)

X(s) – (s +1)Y(s) = -3 (2)

Multiply equation (1) by (s+1) and equation (2) by (3) and add, therefore

$$[s^2+6s+8]\,X(s) = 10(s+1)/s + 2(s+1) + 12$$

Thus $X(s) = \frac{10(s+1)}{s(s+4)(s+2)} + \frac{2s+14}{s^2+6s+8} = (5/2)\left(\frac{1}{s} - \frac{3}{s+4} + \frac{2}{s+2}\right) + \frac{2(s+3)+8}{(s+3)^2-1}$

Hence x(t) = (5/2)(1-3 e^{-4t} +2e^{-2t})+ e^{-3t} (2cosht + 8sinht), but $\frac{dx}{dt}+5x+4y=10$, from which we get y(t). (5 marks)

b- i) Let Y = 1/y = $a+bx+cx^2$, so that the constants can be obtained using the following matrix form :

$$\begin{pmatrix} N & \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 \\ \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^4 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N} Y_i \\ \sum_{i=1}^{N} x_i Y_i \\ \sum_{i=1}^{N} x_i^2 Y_i \end{pmatrix}$$

Given N= 3 , $\sum_{i=1}^{N} x_i$ = 8, $\sum_{i=1}^{N} x_i^2$ = 46, $\sum_{i=1}^{N} x_i^3$ = 242, $\sum_{i=1}^{N} x_i^4$ = 1378, $\sum_{i=1}^{N} Y_i$ = 0.861, $\sum_{i=1}^{N} x_i Y_i$ = 0.92, $\sum_{i=1}^{N} x_i^2 Y_i$ = 6.75, thus by Cramer rule we get a , b and *c* such that: a = $\frac{\Delta_a}{\Delta}$, b = $\frac{\Delta_b}{\Delta}$, c = $\frac{\Delta_c}{\Delta}$,

where $\Delta = \begin{vmatrix} 3 & 8 & 46 \\ 8 & 46 & 242 \\ 46 & 242 & 1378 \end{vmatrix}$, $\Delta_a = \begin{vmatrix} 0.86 & 8 & 46 \\ 0.92 & 46 & 242 \\ 6.75 & 242 & 1378 \end{vmatrix}$, $\Delta_b = \begin{vmatrix} 3 & 0.86 & 46 \\ 8 & 0.92 & 242 \\ 46 & 6.75 & 1378 \end{vmatrix}$,

$$\Delta_c = \begin{vmatrix} 3 & 8 & 0.86 \\ 8 & 46 & 0.92 \\ 46 & 242 & 6.75 \end{vmatrix}$$

(5 marks)

ii) Put y\`= z, hence y\`\`= z\` = x^3 (z + y) = f(x, y, z), such that $x_0 = 0$, $y_0 = 1$, y\`(0) = z_0 = ½.

According to the following two formulas, we can get the first two approximations

$$y_{n+1} = y_0 + \int_{x_0}^{x} z_n\, dx \ , \quad z_{n+1} = z_0 + \int_{x_0}^{x} f(x, y_n, z_n)\, dx = \int_{x_0}^{x} x^3 (y_n + z_n)\, dx$$

Put n=0 to obtain y_1, z_1 such that $y_1 = y_0 + \int_{x_0}^{x} z_0\, dx = 1 + \int_0^x \frac{dx}{2} = 1 + \frac{x}{2}$ and

$$z_1 = z_0 + \int_{x_0}^{x} f(x, y_0, z_0)\ dx = \frac{1}{2} + \int_{x_0}^{x} x^3\ (y_0 + z_0)\ dx = \frac{1}{2} + \int_{0}^{x} x^3\ (1 + \frac{1}{2})\ dx = \frac{1}{2} + \frac{3x^4}{8}$$

Put n=1 to obtain y_2, such that $y_2 = y_0 + \int_{x_0}^{x} z_1\ dx = 1 + \int_{0}^{x} (\frac{3x^4}{8} + \frac{1}{2})\ dx = 1 + \frac{x}{2} + \frac{3x^5}{40}$ (3marks)

By using Euler method, states:

$y_{n+1} = y_n + h\ z_n$, h = 0.1 and $z_{n+1} = z_n + 0.1\ x_n^3\ (y_n + z_n)$

$y_1 = y_0 + h\ z_0 = 1+0.1(1/2) = 1.05$ and $z_1 = z_0 + 0.1\ x_0^3\ (y_0 + z_0) = ½$

$y_2 = y_1 + h\ z_1 = 1.05 + 0.1(1/2) = 1.1 = y(0.2)$ (2 marks)

Answer of Question 4

a-i) $\frac{\partial u}{\partial x} = e^x\ (x \cos y + \cos y - y \sin y)$ and $\frac{\partial u}{\partial y} = e^x(-x \sin y - \sin y - y\cos y)$

Since the function is analytic, therefore Cauchy-Riemann equation is satisfied, hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x\ (x \cos y + \cos y - y \sin y)$$

Integrate the above equation with respect to y, keeping x constant, then

$v = e^x(x \sin y + y\cos y) + \phi(x)$, where $\phi(x)$ is an arbitrary real function of x.

But $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, therefore $e^x(x \sin y + \sin y + y\cos y) + \phi'(x) = e^x(x \sin y + \sin y + y\cos y)$, thus

$\phi'(x) = 0$ and $\phi(x) = c$, c is a constant which we can neglect, therefore $v = e^x(x \sin y + y\cos y)$. If we put x = z and y = 0, thus $f(z) = z\ e^z$. (5 marks)

ii) From Inverse Lagrange formula, we get

$$x = \frac{(y+7)(y-4)(y-47)(y-15)}{(3+7)(3-4)(3-47)(3-15)}(1) + \frac{(y-3)(y-4)(y-47)(y-15)}{(-7-3)(-7-4)(-7-47)(-7-15)}(5)$$

$$+ \frac{(y-3)(y+7)(y-47)(y-15)}{(4-3)(4+7)(4-47)(4-15)}(-13) + \frac{(y-3)(y+7)(y-4)(y-15)}{(47-3)(47+7)(47-4)(47-15)}(2).$$

$$+ \frac{(y-3)(y+7)(y-4)(y-47)}{(15-3)(15+7)(15-4)(15-47)}(-6)$$

Put y = 0, so that we get one of the roots. (5 marks) b- i) Put Ln(2t) = -y, thus dt = - ($e^{-y}/2$) dy, therefore $\int_0^{1/2} t^{m-1} \ln(1/2t)\,dt = \frac{1}{2^m}\int_0^{\infty} y e^{-my}\,dy$

Put my = x, hence $\int_0^{1/2} t^{m-1} \ln(1/2t)\,dt = \frac{1}{2^m}\int_0^{\infty} y e^{-my}\,dy = \frac{1}{2^m}\int_0^{\infty} \frac{x}{m} e^{-x}\,\frac{dx}{m} = \frac{1}{2^m m^2}$

ii) Since z = 5 is outside contour and z = 1 inside contour, therefore:

$$\oint_c \frac{\cos(z)}{z^2-6z+5}\,dz = \oint_c \frac{\cos(z)/(z-5)}{z-1}\,dz = 2\pi i\,(\frac{\cos 1}{-4})$$

iii) $\int_2^{\infty} e^{-x^2+4x-4}\,dx = \int_2^{\infty} e^{-(x-2)^2}\,dx$, Put $y = (x-2)^2 \Rightarrow dy = 2(x-2)\,dx \Rightarrow dx = -\frac{dy}{2\sqrt{y}}$

$$\int_2^{\infty} e^{-(x-2)^2}\,dx = \frac{1}{2}\int_0^{\infty} y^{-1/2} e^{-y}\,dy = \frac{\sqrt{\pi}}{2}$$

iv) $\oint_c \frac{z^3+5z+7}{(z-i)^2}\,dz = = 2\pi i\ f`(i) = 4\pi i$

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