Faculty of Engineering (Shoubra) Benha University Time allowed: 3 hours	1st year Production Mechanical Department	Engineering Mathematics and Physics Department 16th of January 2011

The Examination consists of 5 questions with 15 sub questions (7 points for each sub question)

Question 1

a- Test the following series for convergence:

i) $\sum_{n=1}^{\infty}\frac{5^n+7^n}{3^n+2^n}$ ii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n(2n!)^2}$ iii) $\sum_{n=1}^{\infty}\frac{e^n}{e^{2n}+1}$

b- Find minimal distance of the point (0, 0, -1) from the plane given by $z = 2x - y + 1$

c- Solve the following differential equations:

i) $(y + \ln(x))dx +(x+y^2)\, dy = 0$ ii) $y`` + y = \sec(x)$ iii) $y` = (y/2x) - (xy)^3$

Question 2

a- Given R = (x, y, z) so that $r = |R| = \sqrt{x^2+y^2+z^2}$. Show that $\nabla (r^n) = n\, r^{n-2} R$, for any integer n, then deduce grad (r), grad (r^2), grad (1/r).

b- Define: Sequence - Cauchy sequence – Order and degree of D.E. – Homogeneous function – Homogeneous D.E. (give an example for each)

c- Verify Green's Theorem to evaluate $\int_c xy\,dx + x^2y^3 dy$, where c is the triangle whose vertices (0,0), (1,0), (1,2) with positive orientation.

Question 3

a- Expand the function $f(x, y) = \tan^{-1}(\frac{x+y}{x-y})$ using Taylor series about (0,1)

b- Solve the D.E. $9x^2 y`` - (4+x) y = 0$ using series solution.

c- Solve the following differential equations:

i) $\frac{dy}{dx} = \frac{x+y+3}{x-y+1}$ ii) $y`` + 2y` + 2y = e^x \sin^2(2x)$  iii) $y`` + 5y` + 6y = 2-x+3x^2$

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Question 4

a- Find envelope of the function $f(x, y, t) = \frac{x}{t} + \frac{y}{1-t} = 1$

b- Convert $y` + \phi(x)\,y = \psi(x)\,y^n$ into linear D.E.

c- Evaluate the following integrals

i) $\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

ii) $\int_{(0,0)}^{(1,-2)} (x+3y)dx + (3x-2y)dy$ where the path c is $y^3 + 5x^3 + 3x = 0$

Question 5

a- Determine the critical points and locate any relative minima, maxima and saddle points of function f defined by $f(x, y) = -x^4 - y^4 + 4xy$

b- Find the volume of the parallelepiped spanned by the vectors

$$u = (1,0,2) \qquad v = (0,2,3) \qquad w = (0,1,3)$$

If $F = (x^2y, yz, x + z)$. Find (i) curl curl F; (ii) grad div F

c- Find interval of convergence for the following power series

i) $\sum_{n=1}^{\infty} \frac{3^n}{(n^2+1)(x-2)^n}$ ii) $\sum_{n=1}^{\infty} \frac{n\, e^{-n^2}}{x^n}$

Good Luck
With all best wishes to succeed
Dr. Khaled El Naggar

Model answer

Answer of Question 1

a-i) By ratio test, we get that $\lim\limits_{n\to\infty} \left(\frac{5^{n+1}+7^{n+1}}{3^{n+1}+2^{n+1}}\right)\left(\frac{3^n+2^n}{5^n+7^n}\right) = \lim\limits_{n\to\infty} \frac{7^{n+1}}{3^{n+1}}\left[\frac{(5/7)^{n+1}+1}{1+(2/3)^{n+1}}\right] \frac{3^n}{7^n}\left[\frac{3^n+2^n}{5^n+7^n}\right]$

= 7/3 > 1, therefore the series is divergent. (3 marks)

ii) Let $U_n = \frac{1}{3^n(2n!)^2}$, $\lim\limits_{n\to\infty}\frac{1}{3^n(2n!)^2}=0$, $U_{n+1} = \frac{1}{3^{n+1}((2n+2)!)^2}$, hence $U_n > U_{n+1}$. By using ratio test, we will get that $\sum_{n=1}^{\infty}\frac{1}{3^n(2n!)^2}$ is convergent, so $\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n(2n!)^2}$ is called absolutely convergent. (2 marks)

iii) Since $\int_1^{\infty}\frac{e^n}{e^{2n}+1}dn = (\tan^{-1}e^n)_1^{\infty} = \tan^{-1}\infty - \tan^{-1}e = -\tan^{-1}e$, therefore $\sum_{n=1}^{\infty}\frac{e^n}{e^{2n}+1}$ is convergent.

(2 marks)

b) Let the point on the plane is (x, y, z) and $f(x, y, z) = x^2 + y^2 + (z+1)^2$ is the square of the distance between (0, 0, -1) and (x, y, z), also $\phi(x, y, z) = z - 2x + y = 1$. By applying conditional extrema, we get $f_x = \lambda\,\phi_x$, $f_y = \lambda\,\phi_y$, $f_z = \lambda\,\phi_z$, thus $2x = \lambda(-2)$, $2y = \lambda(1)$ and $2(z+1) = \lambda(1)$, therefore x =-2y and z = y-1. Substitute in $\phi(x, y, z) = z - 2x + y = 1$ so that (-2/3, 1/3, -2/3) is the point on plane and the minimal distance from point (0, 0, -1) $= \sqrt{2/3}$. (7 marks)

c-i) $(y + \ln(x))dx + (x+y^2)\,dy = 0$ is exact D.E. since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$, therefore $\frac{\partial f}{\partial x} = M(x,y) = (y + \ln(x)) \Rightarrow f(x,y) = xy + x\text{Ln}\,x - x + \phi(y)$, thus $\frac{\partial f}{\partial y} = x + \phi'(y) = x+y^2$, hence $\phi(y) =$ y`` + 2y` + 2y = $e^x \sin^2(2x)y^3/3$, therefore solution is $f(x,y) = xy + x\text{Ln}\,x - x + y^3/3 + c$

(3 marks)

ii) y`` + y = sec(x) has homogeneous and particular solution so that the characteristic equation is $m^2 + 1 = 0 \Rightarrow m = -i, i$, thus $y_H = (c_1 \cos x + c_2 \sin x)$ and so the particular solution is:

$y_P = u_1(x)\cos x + u_2(x)\sin x$, and $y_1(x) = \cos x$, $y_2(x) = \sin x$ where $u_1(x) = -\int \frac{y_2 g(x)}{W(y_1, y_2)} dx$,

$u_2(x) = \int \frac{y_1 g(x)}{W(y_1, y_2)} dx$, where $W(y_1, y_2) = y_2' y_1 - y_2 y_1' = 1$, $g(x) = \sec x$, therefore:

$u_1(x) = -\int \frac{\sin x \sec x}{1} dx = \operatorname{Ln} \cos x$ and $u_2(x) = \int \frac{\cos x \sec x}{1} dx = x$ (2 marks)

iii) $y' = (y/2x) - (xy)^3$ is Bernoulli D.E. , therefore $y^{-3} y' - y^{-2}/2x = -x^3$. Put $z = y^{-2}$, therefore $z' = -2 y^{-3} y'$, thus $z' + z/x = 2x^3$ which is linear D.E. whose solution is $zx = -2x^5/5 + c$, hence $xy^{-2} = -2x^5/5 + c$ is the solution of D.E. (2 marks)

Answer of Question 2

a- $\nabla(r^n) = \nabla (x^2 + y^2 + z^2)^{n/2} = \frac{n}{2}(x^2 + y^2 + z^2)^{(n/2)-1}(2x\hat{i} + 2y\hat{j} + 2z k) = n r^{n-2} R$

Put n = 1, therefore grad (r) = $\frac{1}{r} R$,

put n = 2, therefore grad (r^2) = 2 R,

put n = -1, therefore grad (1/r) = $-\frac{1}{r^3} R$. (7 marks)

b- **Sequence:** group of elements related by general term whose domain is set of positive integers.

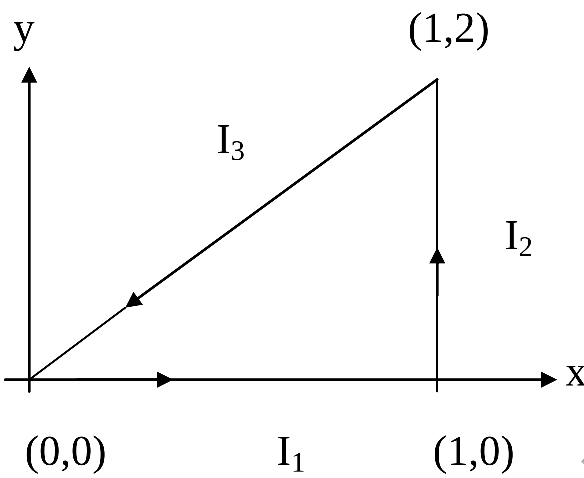
Cauchy Sequence: Every convergent sequence is called Cauchy sequence.

Order of D.E. : is the highest derivative of D.E.

Degree of D.E. : is the power of the highest derivative of D.E.

Homogeneous function: f(x,y) is homogeneous of order n if f(tx, ty) = t^n f(x,y).

Homogeneous D.E. : is the D.E. in which the particular solution equal zero. (7 marks) c-



If we express the problem in line integral, we have to divide the path into three paths $I_1 : y = 0$, $I_2 : x = 1$, $I_3 : 2x = y$ so that $dy = 0$, $dx = 0$, $2dx = dy$ respectively.

For path I_1: $\int_{I_1} xy\,dx + x^2y^3dy = \int_0^1 xy\,dx + x^2y^3dy = 0$

For path I_2: $\int_{I_2} xy\,dx + x^2y^3dy = \int_0^2 y^3dy = 4$

For path I_3: $\int_{I_3} xy\,dx + x^2y^3dy = \int_1^0 [x(2x) + x^2(2x)^3(2)]dx = -\frac{10}{3}$, therefore

$$\int_c xy\,dx + x^2y^3dy = I_1 + I_2 + I_3 = \frac{2}{3}.$$

By using Green theorem

$$\int_c xy\,dx + x^2y^3dy = \iint_D (2xy^3 - x)\,dxdy = \int_{x=0}^{1}\int_{y=0}^{y=2x} (2xy^3 - x)\,dydx$$

$$= \int_{x=0}^{1} (8x^5 - 2x^2)dx = \frac{2}{3}$$ (7 marks)

Answer of Question 3

a- Since $f(x, y) = \tan^{-1}(\frac{x+y}{x-y})$, therefore $f_x = \frac{-y}{x^2+y^2}$, $f_y = \frac{x}{x^2+y^2}$, $f_{xx} = \frac{2xy}{(x^2+y^2)^2}$, $f_{xx} = \frac{2xy}{(x^2+y^2)^2}$, $f_{yy} = \frac{-2xy}{(x^2+y^2)^2}$, and $f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}$. At (0,1), therefore $f(0,1) = -\frac{\pi}{4}$, $f_x = -1$, $f_y = 0$, $f_{xx} = f_{yy} = \frac{2xy}{(x^2+y^2)^2} = 0$, $f_{xy} = 1$, then by substituting in Taylor formula, we get: $f(x, y) = -\frac{\pi}{4} - x + x(y-1)$

b- We note that the series solution at $x = x_0$, $x_0 \neq 0$ still in regular case, so that the solution can be expressed as in (4), but $p(x) = 0$ & $q(x) = \frac{-(4+x)}{9x^2}$ are not analytic at $x = 0$, therefore $x = 0$ is called regular singular since $xp(x)$ & $x^2q(x)$ are analytic at $x = 0$, then series solution about $x = 0$ can be expressed in the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c},$$

$$y'(x) = \sum_{n=0}^{\infty} (n+c)\, a_n x^{n+c-1},$$

$$y''(x) = \sum_{n=0}^{\infty} (n+c)(n+c-1)\, a_n x^{n+c-2}, \ldots,$$

Substitute in the above D.E. , so we get

$$9x^2 \sum_{n=0}^{\infty} (n+c)(n+c-1)\, a_n x^{n+c-2} - (4+x) \sum_{n=0}^{\infty} a_n x^{n+c} = 0$$

$$9 \sum_{n=0}^{\infty} (n+c)(n+c-1)\, a_n x^{n+c} - 4 \sum_{n=0}^{\infty} a_n x^{n+c} - \sum_{n=0}^{\infty} a_n x^{n+c+1} = 0$$

Put n = m-1 for the 3rd t term, n= m for 1st and 2nd terms

We get $[9c(c-1)-4]\, a_0 x^c + \sum_{m=1}^{\infty} ([9(m+c)(m+c-1)-4]a_m - a_{m-1})x^{m+c} = 0$

By comparing of coefficients of x^c, therefore

$[9c(c-1) -4]\ a_0 = 0$, $a_0 \neq 0$, then $9c(c-1) -4= 0$, from which $c_1= -1/3$, $c_2 = 4/3$, therefore $c_1 - c_2 \neq$ integer. (case 1)

By comparing of coefficients of x^{m+c}, therefore $a_m = \dfrac{a_{m-1}}{9(m+c)(m+c-1)-4}$

At c = -1/3

$$a_m = \frac{a_{m-1}}{9m^2 - 15m}, \quad m = 1, 2, \ldots$$

$$a_1 = \frac{-a_0}{6},\ a_2 = \frac{a_1}{6} = \frac{-a_0}{36},\ a_3 = \frac{a_2}{36} = -\frac{a_0}{(36)^2}, \text{ therefore}$$

$$U(x) = x^{-1/3} a_0 \left[1 - \frac{1}{6}x - \frac{1}{36}x^2 - \frac{1}{(36)^2}x^3 + \ldots\right]$$

At c = 4/3 , $a_m = \dfrac{a_{m-1}}{9m^2 + 15m}$, m = 1, 2, ...

$$a_1 = \frac{a_0}{24},\ a_2 = \frac{a_1}{66} = \frac{a_0}{1584},\ a_3 = \frac{a_2}{126} = \frac{a_0}{199584}, \text{ therefore}$$

$$V(x) = x^{4/3} a_0 \left[1 + \frac{1}{24}x + \frac{1}{1584}x^2 + \frac{1}{199584}x^3 + \ldots\right], \text{ thus}$$

$$Y(x) = A a_0 x^{-1/3} \left[1 - \frac{1}{6}x - \frac{1}{36}x^2 - \frac{1}{(36)^2}x^3 + \ldots\right] +$$

$$B\, x^{4/3} a_0 \left[1 + \frac{1}{24}x + \frac{1}{1584}x^2 + \frac{1}{199584}x^3 + \ldots\right]$$ (7 marks)

c- i) Since $\frac{dy}{dx}=\frac{x+y+3}{x-y+1}$ is non homogeneous equation. To solve this differential equation, we have to follow these steps

(1) We have to get the point of intersection between x+y+3=0, x-y+1=0 which is (-2,-1),

(2) Put x=X-2, y=Y-1, dx=dX,dy=dY in the above differential equation, then $\frac{dY}{dX}=\frac{X+Y}{X-Y}$, so it is a homogeneous equation,

(3) Put Y=vX, and dY=vdX+Xdv, therefore $\frac{vdX+Xdv}{dX}=\frac{X+vX}{X-vX}=\frac{1+v}{1-v}$

(4) Integrate $\frac{dX}{X}=\frac{(1-v)dv}{1+v^2}$, then put X=x+2, $v=\frac{Y}{X}=\frac{y+1}{x+2}$ so that the solution of the differential equation is $\text{Ln}(x+2)=\tan^{-1}(\frac{y+1}{x+2})-\frac{1}{2}\ln(\frac{(y+1)^2+(x+2)^2}{(x+2)^2})+C$ (3 marks)

ii) $y'' + 2y' + 2y = e^x \sin^2(2x)$ has homogeneous and particular solution so that the characteristic equation is $m^2 + 2m+2=0 \Rightarrow m = -1 \pm i$, thus $y_H= e^{-x}(c_1\cos x+ c_2 \sin x)$and so the particular solution is $y_P=\frac{1}{D^2+2D+2}e^x\sin^2 x=\frac{1}{D^2+2D+2}(\frac{e^x}{2})(1-\cos 2x)=\frac{e^x}{2}[\frac{1}{5}-\frac{(8\sin 2x+\cos 2x)}{65}]$ (2 marks)

iii) $y'' + 5y' + 6y = 2-x+3x^2$ has homogeneous and particular solution so that the characteristic equation is $m^2 + 5m+6=0 \Rightarrow m = -2, -3$, thus $y_H= c_1e^{-2x} + c_2 e^{-3x}$)and so the particular solution is

$$y_P=\frac{1}{D^2+5D+6}(2-x+3x^2)=\frac{1}{6}(1+\frac{D^2+5D}{6})^{-1}(2-x+3x^2)=\frac{1}{6}(6-6x+3x^2)$$ (2 marks)

Answer of Question 4

a- Differentiate w.r.t. t such that $-x/t^2 - y(1-t)^{-2}(-1) = 0$, therefore $t=\frac{1}{1\pm\sqrt{\frac{y}{x}}}$, and $1-t=\frac{\pm\sqrt{\frac{y}{x}}}{1\pm\sqrt{\frac{y}{x}}}$, thus the envelope is $(1\pm\sqrt{\frac{y}{x}})(x\pm\sqrt{xy})=1$ (7 marks) b- A differential equation of Bernoulli type is written as $y' + \phi(x)\,y = \psi(x)\,y^n$

This type of equation is solved via a substitution. Indeed, let $z = y^{1-n}$, then easy calculations give

$z' = (1-n)\,y^{-n}\,y'$ which implies $\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$. (7 marks)

c- i) Put $x = r\cos\theta$, $y = r\sin\theta$, therefore $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{2}\int_0^{\pi} r^2\,dr\,d\theta = \frac{8}{3}\pi$ (3 marks)

ii) Since $P_y = 3 = P_x$, therefore the integral independent on the path, thus we can take the path as the line joining between the two end points such that y = -2x, hence dy = -2dx.

Therefore $\int_{(0,0)}^{(1,-2)} (x+3y)dx + (3x-2y)dy = \int_0^1 -19x\,dx = -19/2$

Answer of Question 5

a- $f_x = -4x^3 + 4y = 0$, $f_y = -4y^3 + 4x = 0$, therefore $y = x^3$, substitute in one of the two equations, hence (0,0), (1,1), (-1,-1) are the critical points and $f_{xx} = -12x^2$, $f_{yy} = -12y^2$, $f_{xy} = 4$.

At (0,0), $\Delta = -16 < 0$, saddle point

At $(\pm 1, \pm 1)$, $\Delta = 128 > 0$, $f_{xx}, f_{yy} < 0$, maximum point (7 marks)

b- Volume = $u.(v \times w) = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & 3 \end{vmatrix} = 3$.

Curl F = $\nabla \times F = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & yz & x+z \end{vmatrix}$ = $-y\,\hat{i} - \hat{j} - x^2\,k$

Curl Curl F = $\begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & -1 & -x^2 \end{vmatrix} = 2x\ \hat{j} + k$

Div F = 2xy + z + 1 , grad div F = $2y\ \hat{i} + 2x\ \hat{j} + k$ (7 marks)

c- i) Since $U_n = \frac{3^n}{(n^2+1)(x-2)^n}$, and $U_{n+1} = \frac{3^{n+1}}{((n+1)^2+1)(x-2)^{n+1}}$, hence the ratio $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{3^{n+1}(n^2+1)(x-2)^n}{3^n((n+1)^2+1)(x-2)^{n+1}}\right| = \left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right|$,therefore

$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right| = \lim_{n\to\infty}\left|\frac{3}{(x-2)}\right| < 1$ to be convergent, hence $|x-2| > 3$, thus x > 5 or x < -1 is the interval of convergence. **(4marks)**

ii) Since $U_n = \frac{n\,e^{-n^2}}{x^n}$, and $U_{n+1} = \frac{(n+1)\,e^{-(n+1)^2}}{x^{n+1}}$, hence the ratio

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{x^n(n+1)\,e^{-(n+1)^2}}{x^{n+1}\ \ n\,e^{-n^2}}\right| = \left|\frac{(n+1)}{ne^{(2n+1)}x}\right|$$

Therefore $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{ne^{(2n+1)}x}\right| = 0 < 1$, hence $\sum_{n=1}^{\infty}\frac{n\,e^{-n^2}}{x^n}$ is convergent for all x. **(3 marks)**

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