Benha University
Faculty of Engineering- Shoubra
Survey Engineering Department
1st Year Survey

Final Term Exam   Date:29-12-2014
Mathematics 2A Code : EMP160
Duration : 3 hours

- Answer all the following questions
- The exam. Consists of one page
- No. of questions: 3
- Total marks: 100

Question 1 **(30 marks)**

a) Test the following series for convergence:

i) $\sum_{n=1}^{\infty} \frac{5^n + 7^n}{3^n + 2^n}$ ii) $\sum_{n=1}^{\infty} \frac{n^n}{(3n+2)^n}$ iii) $\sum_{n=1}^{\infty} \frac{4n^2 + n}{\sqrt[3]{n^7 + n^3}}$ iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5 + n}$

b) Find interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-6)^n}{n\, 3^n}$

Question 2 **(30 marks)**

i) If s=f(u,v) = $u^2 + v^3$, $u = x^2$ and v = sinx, find $\frac{ds}{dx}$

ii) Find envelope of the function $(x-\cos\alpha)^2 + (y-\sin\alpha)^2 = \rho$

iii) Expand the function $f(x, y) = \ln(\frac{x+y}{x-y})$ using Taylor series about (0,1)

iv) $u = \tan^{-1}[\frac{x^3 + y^3}{2x + 3y}]$, show that $x\, u_x + y\, u_y = \sin 2u$

v) Find all relative extrema and saddle points for $f(x, y) = -x^2 - 4x - y^2 + 2y - 1$

Question 3 **(40 marks)**

Solve the following differential equations:

a) $y` = \frac{\cos y - ye^x}{e^x + x\sin y}$    **b)** y` = (y/x) + tan(y/x)   **c)** $y` = (y/2x) - (xy)^3$

d) $y'' + 5y' + 4y = e^{5x} \cos 2x$ **e)** $y`` - 2y` + y = (x^2 - 1)\, e^{2x} + (3x+4)\, e^x$

Questions	Total marks	Achieved ILOS
Q1	30	b1
Q2	30	a1
Q3	40	a1, c1

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Model answer

1a-i) By ratio test, we get that $\lim_{n\to\infty} (\frac{5^{n+1}+7^{n+1}}{3^{n+1}+2^{n+1}})(\frac{3^n+2^n}{5^n+7^n}) = \lim_{n\to\infty} \frac{7^{n+1}}{3^{n+1}}[\frac{(5/7)^{n+1}+1}{1+(2/3)^{n+1}}]$

$\frac{3^n}{7^n}[\frac{3^n+2^n}{5^n+7^n}]$ = 7/3 > 1, therefore the series is divergent.

1a-ii) $\lim_{n\to\infty} \sqrt[n]{[\frac{n}{(3n+2)}]^n} = \lim_{n\to\infty}[\frac{n}{(3n+2)}] = \frac{1}{3} < 1$, thus $\sum_{n=1}^{\infty} \frac{n^n}{(3n+2)^n}$ is convergent.

1a-iii) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n^{-1/3}$ is divergent, therefore $\sum_{n=1}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ is divergent.

1a-iv) Since $\lim_{x\to\infty} \frac{1}{n^5+n} = 0$, $U_n > U_{n+1}$, and $\sum_{n=1}^{\infty} \frac{1}{n^5+n}$ is convergent

1-b) Since $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(-1)^{n+1}(x-6)^{n+1}[n\,3^n]}{(-1)^n(x-6)^n[(n+1)\,3^{n+1}]}\right| = \left|-\frac{(x-6)^{n+1}[n\,3^n]}{(x-6)^n[(n+1)\,3^{n+1}]}\right|$ thus

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(x-6)^{n+1}[n\,3^n]}{(x-6)^n[(n+1)\,3^{n+1}]}\right| = \left|\frac{x-6}{3}\right| < 1 \Rightarrow |x-6| < 3 \Rightarrow \quad 3 < x < 9$$

2-i) $\frac{ds}{dx} = \frac{\partial s}{\partial u}\frac{du}{dx} + \frac{\partial s}{\partial v}\frac{dv}{dx} = 2u(2x) + 3v^2(\cos x)$

ii) $\frac{\partial}{\partial\alpha}[(x-\cos\alpha)^2 + (y-\sin\alpha)^2 = \rho] \Rightarrow 2(x-\cos\alpha)\sin\alpha = 2(y-\sin\alpha)\cos\alpha \Rightarrow \tan\alpha = y/x$, therefore $-x\sin\alpha + y\cos\alpha = 0$, thus $\tan\alpha = y/x$, so $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$, hence envelope is $\sqrt{x^2+y^2} - 1 = \pm\rho$.

iii) We have to get f_x, f_y, f_{xx}, f_{xy}, f_{yy} such that $f_x = \frac{1}{x+y} - \frac{1}{x-y}$, $f_y = \frac{1}{x+y} + \frac{1}{x-y}$,

$f_{xx} = \frac{-1}{(x+y)^2} + \frac{1}{(x-y)^2}$, $f_{yy} = \frac{-1}{(x+y)^2} + \frac{1}{(x-y)^2}$, $f_{xy} = \frac{-1}{(x+y)^2} - \frac{1}{(x-y)^2}$

Therefore: at (0, 1), f (0, 1) = 0, f_x = 2, f_y = 0, f_{xx} =0, f_{yy} = 0, f_{xy} = -2, therefore f(x,y)= f(0, 0) + $\frac{1}{1!}$ ($f_x(0,0)$ (x-0) + $f_y(0,0)$(y-0)) + $\frac{1}{2!}$ ($f_{xx}(0,1)(x-0)^2$ + 2(x-0) (y-1) $f_{xy}(0, 1)$ + $f_{yy}(0, 1) (y-1)^2$) , therefore f(x,y) = 2x + x (y-1)

iv) Since $\frac{x^3 + y^3}{2x + 3y}$ = tan u = z is homogenous of degree 2, therefore x z_x + y z_y = 2 z $\Rightarrow$

$x \frac{dz}{du}\frac{\partial u}{\partial x} + y \frac{dz}{du}\frac{\partial u}{\partial y} = 2\tan u$, hence x u_x + y u_y = sin2u.

v) Find the first partial derivatives f_x and f_y such that $f_x(x,y) = -2x - 4$, $f_y(x,y) = -2y + 2$

Determine the critical points by solving the equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$ simultaneously, hence, -2x – 4 = 0 , - 2y + 2 = 0, therefore the critical point is (-2,1), then determine the second order partial derivatives such that: $f_{xx}(x,y) = -2$, $f_{yy}(x,y) = -2$, $f_{xy}(x,y) =0$, hence $\Delta > 0$, therefore it is a maximum point.

3-a) $(\cos y - ye^x)dx - (e^x + x\sin y)dy = 0 \Rightarrow$ $M_y = N_x$= -siny – $e^x \Rightarrow$ therefore the D.E. is exact, thus

f_x= cosy - y $e^x \Rightarrow$ f(x, y) = xcosy - y e^x + g(y) $\Rightarrow$ f_y = -x siny - e^x + g\`(y) = N(x,y) $\Rightarrow$ g\`(y) = 0, therefore g(y) = c $\Rightarrow$ f(x, y) = xcosy - y e^x + c

3-b) Put y = vx $\Rightarrow$ dy = vdx + xdv $\Rightarrow$ vdx + xdv = (v + tanv)dx $\Rightarrow$ cotv dv = dx/x $\Rightarrow$ lnsin (y/x) = lnx

3-c) $y' = (y/2x) - (xy)^3$ is Bernoulli D.E., thus $y^{-3}y' - y^{-2}/2x = -x^3$. Put $z = y^{-2} \Rightarrow z' = -2y^{-3}y' \Rightarrow$ $z' + z/x = 2x^3$ which is linear D.E. whose solution is $zx = -2x^5/5 + c$, so $xy^{-2} = -2x^5/5 + c$ is the solution of D.E.

3-d) The characteristic equation is $r^2 + 5r + 4 = 0$, therefore r=-4, -1, thus $y_c = (c_1 e^{-4x} + c_2 e^{-x})$

$$Y_P = \frac{1}{D^2 + 5D + 4} e^{5x}\cos 2x = e^{5x}\frac{1}{(D+5)^2 + 5(D+5) + 4}\cos 2x = e^{5x}\frac{1}{D^2 + 15D + 54}\cos 2x$$

$$= e^{5x}\frac{1}{15D + 50}\cos 2x = e^{5x}\frac{(15D - 50)}{225D^2 + 2500}\cos 2x = e^{5x}\frac{(-30\sin 2x - 50\cos 2x)}{225(-4) + 2500}$$

3-e) The characteristic equation is $r^2 - 2r + 1 = 0$, therefore r=-1, -1, thus $y_c = (c_1 e^{-x} + c_2 xe^{-x})$

$$Yp(x) = \frac{1}{D^2 - 2D + 1}[(x^2 - 1)e^{2x} + (3x+4)e^x] = e^{2x}\frac{1}{D^2 + 2D + 1}(x^2 - 1) - e^x\frac{1}{D^2}(3x+4)$$

$= e^{2x}[1 - 2D + 3D^2](x^2 - 1) - e^x(x^3/2 + 2x^2)$