Benha University Faculty of Engineering – Shoubra Department of Energy and Sustainable Energy Course: Mathematics 3 Code: EMP 201		Final Exam Date: January 24, 2016 Duration: 3 hours Answer **All** questions
The exam consists of one page	No. of questions: 4	Total Mark: 40

Question 1

(a)Find the first and second derivatives of the function : 3

$f(x,y,z) = x^4 2^y + \sin xy + z \ln z$

(b)Find the envelope of the curves : $x^2 + (y-a)^2 = 2a$. 2

(c)If $u = \sin^{-1}\frac{xz}{x^2+2y^2} + \operatorname{sech}^{-1}\frac{yz}{y^2-3z^2}$. 2

Show that $x.u_x + y.u_y + z.u_z = 0$

(d)Determine the extrema of the function : $f(x,y) = x^3 - y^3 + 3xy$. 3

Question 2

(a)Find a point on $x^2 + y^2 - 12x + 35 = 0$ which is nearest the origin. 2

(b)Find the area of the region bounded by the curve: 2

$x = t+1,\ y = 1 + e^{2t},\ \ 0 \le t \le 1.$

(c)Find the length of the curve: $x = \frac{1}{2}\ln(1+t^2),\ y = \tan^{-1} t,\ 0 \le t \le 1$ 3

(d)Find the surface area of the surface generated by rotating, about x-axis, 3

the curve $x = 2t,\ \ y = \ln t - \frac{1}{2}t^2,\ \ 1 \le t \le 2.$

Question 3

(a)Find the unit vector in the direction normal to $x^2y + 2xz = 4$ at the point $(2,-2,3)$ 2

(b)Show that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is conservative field, find the scalar potential for this field and find the work done due to move body from $(1,-2,1)$ to $(3,1,4)$ in this field. 4

(c)Evaluate $\oiint \overline{F}.\overline{n}ds$ given that $\overline{F} = 2yx\overline{i} + yz^2\overline{j} + xz\overline{k}$ and S is the surface of parallelogram bounded by : 4

$x = 0,\ y = 0,\ z = 0,\ x = 2,\ y = 1,\ z = 3.$

Question 4

(a)If $\vec{F} = (x^2yz)\vec{i} + (xyz)\vec{j} - (xyz^2)\vec{k}$. Find div $\vec{F}$ and curl $\vec{F}$. 3

(b)Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F}(t) = (x-3y)\vec{i} + (y-2x)\vec{j}$ and C is the 3

closed curve $x = 3\cos\theta,\ y = 2\sin\theta$ in xy – plane. (c)Evaluate $\iint_S (\vec{\nabla}\times F).\vec{n}\,dS$ given that $\vec{F} = (2x-y)\vec{i} - (yz^2)\vec{j} - (y^2 z)\vec{k}$ over the surface of the sphere : $x^2 + y^2 + z^2 = 1,\ z \geq 0$.	4

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Answer

Answer of Question 1

(a) $f_x = 4x^3 2^y + \cos xy\,.y,\quad f_y = x^4 2^y.\ln 2 + \cos xy\,.x,\quad f_z = z.\frac{1}{z} + \ln z$

$f_{xx} = 12x^2 2^y - \sin xy\,.y^2,\ f_{yy} = x^4 2^y (\ln 2)^2 - \sin xy\,.x^2,\ f_{zz} = 0 + \frac{1}{z}$

$f_{xy} = 4x^3 2^y.\ln 2 - \sin xy\,.xy + \cos xy\,.1,\quad f_{xz} = 0,\quad f_{yz} = 0$

--*3-Marks*

(b)Differentiate with respect to a, we get $0 - 2(y-a) = 2$

Then a = y + 1.

Then the envelope is : $x^2 + (-1)^2 = 2(y+1)$ Or $x^2 = 2y + 1$

--*2-Marks*

(c)u(x, y, z) is homogenous of degree 0.

Then from Euler;s theorem, we get $x.u_x + y.u_y + zu_z = 0$.

--*2-Marks*

(d)Since $f_x = 3x^2 + 3y = 0,\quad f_y = -3y^2 + 3x = 0$

Then $x = y^2$ and $3y^4 + 3y = 0$. Then y = 0, – 1.

Then, we get the points $P_1 = (0,0),\ P_2 = (1,-1)$.

At $P_1 = (0,0)$, $\Delta = f_{xx}.f_{yy} - (f_{xy})^2 = (6x)(-6y) - 9 = -9$, Saddle point.

At $P_2 = (1,-1)$, $\Delta = f_{xx}.f_{yy} - (f_{xy})^2 = (6x)(-6y) - 9 = 27$, minimum point.

--*3-Marks*

Answer of Question 2

(a) The distance is : $d = \sqrt{x^2 + y^2}$. Then $f(x,y) = x^2 + y^2$

From the equation : $x^2 + y^2 = \lambda(x^2 + y^2 - 12x + 35)$

$\frac{\delta}{\delta x}$: $2x = \lambda(2x - 12)$, then

$\frac{\delta}{\delta y}$: $2y = \lambda(2y)$, then $2y(1 - \lambda) = 0$ and $y = 0$ or $\lambda = 1$

If $\lambda = 1$, the first equation is not satisfied.

Then $\lambda \neq 1$, $y = 0$, substitute in the given equation, we get

$x^2 - 12x + 35 = (x - 5)(x - 7) = 0$

Then we get, x = 5, x = 7 and the points $P_1 = (5,0)$, $P_2 = (7,0)$

Since $f(5,0) = 25$ *and* $f(7,0) = 49$. Then the required point is $P_1 = (5,0)$

---*2-Marks*

$$(b)\ A = \int_0^1 y.\dot{x}\,dt = \int_0^1 (1 + e^{2t})\,dt = t + \frac{1}{2}e^{2t} = \frac{1}{2} + \frac{1}{2}e^2$$

---*2-Marks*

$$(c)\ L = \int_0^1 \sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt = \int_0^1 \sqrt{[\frac{t}{1+t^2}]^2 + [\frac{1}{1+t^2}]^2}\,dt$$

$$= \int_0^1 \sqrt{\frac{t^2+1}{(1+t^2)^2}}\,dt = \int_0^1 \frac{1}{\sqrt{1+t^2}}\,dt = \sinh^{-1} t = \sinh^{-1} 1 = 0.84$$

---*3-Marks*

$$(d)\ S_x = 2\pi \int_1^2 y.\sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt = 2\pi \int_1^2 (\ln t - \frac{1}{2}t^2).\sqrt{4 + [\frac{1}{t} - t]^2}\,dt$$

$$= 2\pi \int_1^2 (\ln t - \frac{1}{2}t^2).\sqrt{(\frac{1}{t} + t)^2}\,dt = 2\pi \int_1^2 (\ln t - \frac{1}{2}t^2).(\frac{1}{t} + t)\,dt$$

$$= 2\pi \int_1^2 (\frac{\ln t}{t} + t\ln t - \frac{1}{2}t - \frac{1}{2}t^3)\,dt$$

$$= 2\pi \left(\frac{1}{2}(\ln t)^2 + \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 - \frac{1}{4}t^2 - \frac{1}{8}t^4\right)$$

$$= 2\pi \left(\left(\frac{1}{2}(\ln 2)^2 + 2\ln 2 - 4\right) - \frac{5}{8}\right)$$

---*3-Marks*

Dr. Mohamed Eid

Answer of Question 3

(a)Find the unit vector in the direction normal to $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

Solution:

Since $\overline{\nabla}\phi$ is a normal to the surface ϕ then the unit vector normal to ϕ in the form $\frac{\overline{\nabla}\phi}{|\overline{\nabla}\phi|}$.Now:

$$\overline{\nabla}\phi = \overline{\nabla}(x^2y + 2xz)$$

$$= \frac{\partial}{\partial x}(x^2y + 2xz)\bar{i} + \frac{\partial}{\partial y}(x^2y + 2xz)\bar{j} + \frac{\partial}{\partial z}(x^2y + 2xz)\bar{k}$$

$$= (2xy + 2z)\,\bar{i} + x^2\,\bar{j} + (2x)\,\bar{k} = -2\,\bar{i} + 4\,\bar{j} + 4\,\bar{k} \text{ at } (2,-2,3).$$

$$\therefore \bar{n} = \frac{\overline{\nabla}\phi}{|\overline{\nabla}\phi|} = \frac{-2\,\bar{i} + 4\vec{j} + 4\bar{k}}{\sqrt{4+16+16}} = \frac{-1}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$$

--*2-Marks*

(b)Show that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is conservative field, find the scalar potential for this field and find the work done due to move body from $(1,-2,1)$ to $(3,1,4)$ in this field.

Solution:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = 0$$

Then the field $\vec{F}$ is conservative and there exist scalar function φ such that $\vec{F} = \vec{\nabla}\varphi$ thus: $F\,dr = \nabla\varphi\,dr = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi$

$$\therefore d\varphi = F.dr = (2xy + z^3)dx + x^2 dy + 3xz^2\, dz$$

$$= (2xydx + x^2 dy) + (z^3 dx + 3xz^2 dz)$$

$$= d(x^2 y) + d(xz^3) = d(x^2 y + xz^3)$$

$\therefore \varphi = x^2 y + xz^3 + c$ c is a constsnt

and the work don in the form:

$$W = \int_{p_1}^{p_2} \vec{F}.d\vec{r} = \int_{p_1}^{p_2} (2xy + z^3)dx + x^2 dy + 3xz^2 dz = \int_{p_1}^{p_2} d\varphi$$

$$= x^2 y + xz^3 \Big|_{p_1}^{p_2} = x^2 y + xz^3 \Big|_{(1,-2,1)}^{(3,1,4)} = 202$$

---*4-Marks*

(c)Evaluate $\iint_S \vec{F}.\vec{n}\, ds$ given that $\vec{F} = 2yx\vec{i} + yz^2\vec{j} + xz\vec{k}$ and S is the surface of parallelogram bounded by $x = 0, y = 0, z = 0, x = 2. y = 1, z = 3$.

Answer

Use divergent theorem $\iint_S \vec{F}.\vec{n}\, ds = \iiint_V \vec{\nabla}.\vec{F}\, dV$

$$\vec{\nabla}.\vec{F} = \frac{\partial}{\partial x} 2yx + \frac{\partial}{\partial y} y\, z^2 + \frac{\partial}{\partial z} xz = 2y + z^2 + x$$

$$\iint_S \vec{F}.\vec{n}\, ds = \iiint_V \nabla.F\, dV = \int_0^2 \int_0^1 \int_0^3 \left(2y + z^2 + x\right) dx dy dz$$

$$= \int_0^1 \int_0^3 \left(2yx + z^2 x + \frac{x^2}{2}\right)_{x=0}^{x=2} dy dz = \int_0^1 \int_0^3 \left(4y + 2z^2 + 2\right) dy dz$$

$$= \int_0^3 \left(4\frac{y^2}{2} + 2z^2 y + 2y\right)_{y=0}^{y=1} dz = \int_0^3 \left(2 + 2z^2 + 2\right) dz$$

$$= \left(2z + 2\frac{z^3}{3} + 2z\right)_{z=0}^{z=3} = \left(6 + 18 + 6\right)_{z=0}^{z=3} = 30$$

--*4-Marks*

<u>Answer of Question 4</u>

(a)If

$\vec{F} = x^2 yz\, \vec{i} + xyz\, \vec{j} - xyz^2\, \vec{k}$ Find $div\ \vec{F},\quad curl\ \vec{F}$

<u>Solution</u>:

$$div\ \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$div\ \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(x^2 yz\right) + \frac{\partial}{\partial y}\left(xyz\right) + \frac{\partial}{\partial z}\left(-xyz^2\right) = 2xyz + xz - 2xyz = xz$$

$$curl\ \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$=\left[\frac{\partial}{\partial y}\left(-xyz^2\right)-\frac{\partial}{\partial z}(xyz)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(x^2yz\right)-\frac{\partial}{\partial x}\left(-xyz^2\right)\right]\vec{j}$$

$$+\left[\frac{\partial}{\partial x}(xyz)-\frac{\partial}{\partial y}\left(x^2yz\right)\right]\vec{k}$$

$$=\left[\left(-xz^2\right)-(xy)\right]\vec{i}+\left[\left(x^2y\right)-\left(-yz^2\right)\right]\vec{j}+\left[(yz)-\left(x^2z\right)\right]\vec{k}$$

$$=\left(-xz^2-xy\right)\vec{i}+\left(x^2y+yz^2\right)\vec{j}+\left(yz-x^2z\right)\vec{k}$$

---*3-Marks*

(b)Evaluate $\oint_C \vec{F}.d\vec{r}$ where $\vec{F}(t)=(x-3y)\vec{i}+(y-2x)\vec{j}$ and C is the closed curve $x=3\cos\theta,\ y=2\sin\theta$ in xy $-$plane

Answer

$x=3\cos\theta,\quad dx=-3\sin\theta d\theta \qquad y=2\sin\theta,\ dy=2\cos\theta d\theta$

$$\oint_C \vec{F}.d\vec{r}=\int (x-3y)dx+(y-2x)dy$$

$$=\int_0^{2\pi}(3\cos\theta-6\sin\theta)(-3\sin\theta d\theta+(2\sin\theta-6\cos\theta)2\cos\theta d\theta$$

$$=\int_0^{2\pi}(-9\cos\theta\sin\theta+18\sin^2\theta+4\sin\theta\cos\theta-12\cos^2\theta)d\theta$$

$$=\int_0^{2\pi}\left[-5\sin\theta\cos\theta+9(1-\cos 2\theta)-6(1+\cos 2\theta)\right]d\theta$$

$$=\left[\frac{5}{2}\sin^2\theta+9\left(\theta-\frac{1}{2}\sin 2\theta\right)-6\left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi}$$

$$=\left[9(2\pi)-6(2\pi)\right]=6\pi$$

Another Solution

Apply Green Theorem

$$\oint_C \vec{F}.d\vec{r} = \int (x-3y)dx + (y-2x)dy$$

$$= \iint_{(x^2/9)+(y^2/4)=1} \left[\frac{\partial(y-2x)}{\partial x} - \frac{\partial(x-3y)}{\partial y}\right] dxdy \quad -$$

$$= \iint_{(x^2/9)+(y^2/4)=1} (-2+3)dxdy = 2\times 3\pi = 6\pi$$

--*4-marks*

(c)Evaluate $\therefore \iint_S (\vec{\nabla}\times F).\vec{n}dS$ given that

$\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ over the surface of the sphere

$x^2 + y^2 + z^2 = 1, \quad z \geq 0$.

Solution

The bound of the surface at $z=0$ is the curve $x^2 + y^2 = 1$

Then $\vec{F} = (2x-y)\vec{i}$ and by using parametric equations

$x = \cos t, \quad y = \sin t$ then $dx = -\sin t dt$

Apply Stokes' theorem

$$\therefore \iint_S (\vec{\nabla}\times F).\vec{n}\,dS = \oint_C \vec{F}.d\vec{r} = \oint_C (2x-y)dx$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt$$

$$= \int_0^{2\pi} (-2\cos t \sin t + \sin^2 t)dt$$

$$= \int_0^{2\pi} \left(-2\cos t \sin t + \frac{1}{2}(1-\cos 2t)\right)dt$$

$$= \cos^2 t + \frac{1}{2}t - \frac{1}{4}\sin 2t \;\Big|_0^{2\pi} = \pi$$

$$\oint_C \vec{F}.d\vec{r} = \oint_C (2x-y)dx = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt$$

$$= \int_0^{2\pi} (-2\cos t \sin t + \sin^2 t)dt$$

$$= \int_0^{2\pi} \left(-2\cos t \sin t + \frac{1}{2}(1-\cos 2t)\right)dt$$

$$= \cos^2 t + \frac{1}{2}t - \frac{1}{4}\sin 2t \;\Big|_0^{2\pi} = \pi \quad ....(1)$$

<u>Another Solution</u>

First we find $\vec{\nabla}\times\overline{F}$ as following $\vec{\nabla}\times\overline{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = \vec{k}$

$$\therefore \iint_S (\vec{\nabla}\times F).\vec{n}\,dS = \iint_S \vec{k}.\vec{n}\,ds = \iint_R dx\,dy = \pi$$

Where $\vec{k}.\vec{n}ds = dxdy$ and the value of the integral $\iint_R dxdy$ is the area of the circle $x^2 + y^2 = 1$ which is equal π

--*4-Marks*

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