Benha University
Faculty of Engineering- Shoubra
Eng. Mathematics & Physics Department
Qualifying Courses (Mathematics)

Final Term Exam
Date: January 16, 2016
Course: Linear Algebra EMM 402
Duration: **3** hours

- Answer **All** questions   The exam consists of one page
- No. of questions: 5   Total Mark: 200

Question 1

(a)Determine the linearly independent and linearly dependent: 10

(i) u = (2, 3), v = (1, 2)   (ii) u = (2, 0, 3), v = (1, 2, 0), w = (3, 2, 3)

(b) If $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 2 \\ 0 & -2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \\ 3 & 2 & 0 \end{bmatrix}$ 40

Find, if possible, A + B, $A + B^t$, A.B, $A.B^t$, A.C, |A| and |C|.

Question 2

(a)If $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}$. 40

(i)Show that A is symmetric and find its eigenvalues and eigenvectors.

(ii)Show that its eigenvectors are orthogonal.

(b) Show that the eigenvalues of : $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ are real numbers, where a, b, c are real numbers. 10

Question 3

Write the following expressions in matrix form and determine the type: 30

(a) $P = 4x^2 + 3y^2 + 2z^2 + 2xy - 2xz + yz$

(b) $P = 4xy + 2xz - 2yz - 2x^2 - 3y^2 - 2z^2$

(c) $P = 2xy - 6xz + 2yz + x^2 + 2y^2 + 3z^2$

Question 4

(a)Write the Hessian matrix of : $f(x, y, z) = xye^z + \cos y + x^3 \ln z$. 10

(b) Find $f(A) = \frac{170A}{A^2+I}$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ 20

Question 5

(a)Write the equations: $a_{11}x + a_{12}y + a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$, $a_{31}x + a_{32}y + a_{33}z = b_3$ in matrix form and discuss the types of solutions. 20

(b)Solve the linear system :

x – 2y + 2z = 2, 2x + y – z = 4, 2x + 2y – z = 6, 3x – 2z = 2 20

Good Luck   *Dr. Mohamed Eid*

Answer

Answer of Question 1

(a)(i) au + bv = a(2, 3) + b(1, 2) = (2a + b, 3a + 2b) = 0

Then 2a + b = 0, 3a + 2b = 0. Then a = b = 0.

Then u and v are linearly independent.

(ii) au + bv + cw = a(2, 0, 3) + b(1, 2, 0) + c(3, 2, 3)

= (2a + b + 3c, 0a + 2b+ 2c, 3a + 0b + 3c) = 0

Then 2a + b +3c = 0, 2b + 2c = 0, 3a + 3c = 0. Then a = b = – c.

Then u, v and w are linearly dependent.

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(b) $A + B = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -2 \end{bmatrix}$

$A + B$`, A.B , |A| are not exist.

$$A.B` = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & -3 \end{bmatrix} . \begin{bmatrix} 2 & 0 \\ -3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -8 & -6 \\ -2 & -3 \end{bmatrix}$$

$$A.C = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & -3 \end{bmatrix} . \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 2 & -4 \\ -7 & -6 & -4 \end{bmatrix}$$

$|C| = 2 - 0 + 18 = 20$

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Answer of Question 2

(a)(i)Since $A` = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix} = A$. Then, it is symmetric.

The characteristic equation is $|A - \lambda I| = 0$

Then $\begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 4-\lambda & 0 \\ -2 & 0 & 5-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)(5-\lambda) - 4(4-\lambda) = 0$

$(4-\lambda)[(2-\lambda)(5-\lambda) - 4] = (4-\lambda)(\lambda - 1)(\lambda - 6) = 0$

Then the eigenvalues are 1, 4, 6.

From the linear system of equations:

$$\begin{bmatrix} 2-\lambda & 0 & -2 \\ 0 & 4-\lambda & 0 \\ -2 & 0 & 5-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

For $\lambda = 1$, we get the system of equations:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$

Then $x + 0y - 2z = 0$, $0x + 3y + 0z = 0$, $-2x + 0y + 4z = 0$

From the second equation $y = 0$.

From the first equation $x = 2z$ = any number, put $z = 1$.

Then the corresponding eigenvector is $X_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 4$, we get the system of equations: $\begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Then $-2x + 0y - 2z = 0$, $0x + 0y + 0z = 0$, $-2x + 0y + z = 0$

From the first equation y = any number.

From the first and third equations $x = z = 0$. Putting $y = 1$.

Then the corresponding eigenvector is $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 6$, we get the system of equations:

$$\begin{bmatrix} -4 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$

Then $-4x + 0y - 2z = 0$, $0x - 2y + 0z = 0$, $-2x + 0y - z = 0$

From the second equation $y = 0$.

From the first or third equations $z = -2x$ = any number, putting $x = 1$.

Then the corresponding eigenvector is $X_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

The matrix of eigenvectors is $T = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

(ii)We see that $X_1`X_2 = 0 = X_1`X_3 = X_2`X_3$

Then, the three eigenvectors are orthogonal.

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(b) Since $|A - \lambda I| = \begin{vmatrix} a-\lambda & c \\ c & b-\lambda \end{vmatrix} = (a-\lambda)(b-\lambda) - c^2 = 0$

Then, the characteristic equation is $\lambda^2 - (a+b)\lambda + ab - c^2 = 0$

The root of this equation are real if $(a+b)^2 \geq 4(ab - c^2)$

Then $a^2 + b^2 + 2ab \geq 4ab - 4c^2$

This inequality is true because $a^2 + b^2 - 2ab + 4c^2 \geq 0$

Or $(a-b)^2 + 4c^2 \geq 0$

Then the eigenvalues are real numbers

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<u>Answer of Question 3</u>

(a) $P = X`AX = [x \quad y \quad z] \begin{bmatrix} 4 & 1 & -1 \\ 1 & 3 & \frac{1}{2} \\ -1 & \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is positive definite.

(b) $P = X`AX = [x \quad y \quad z] \begin{bmatrix} -2 & 2 & 1 \\ 2 & -3 & -2 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is negative definite.

(C) $P = X`AX = [x \quad y \quad z] \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is indefinite.

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Answer of Question 4

(a) $H = \begin{bmatrix} 6x\ln z & e^z & \frac{3x^2}{z} \\ e^z & -\cos y & xe^z \\ \frac{3x^2}{z} & xe^z & xye^z \end{bmatrix}$

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(b) The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4.$

$$p(\lambda) = f(\lambda_1)\frac{(\lambda-\lambda_2)(\lambda-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + f(\lambda_2)\frac{(\lambda-\lambda_1)(\lambda-\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + f(\lambda_3)\frac{(\lambda-\lambda_1)(\lambda-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$= \frac{85}{6}(\lambda-3)(\lambda-4) - \frac{51}{2}(\lambda-1)(\lambda-4) + \frac{40}{3}(\lambda-1)(\lambda-3)$$

$$= 2\lambda^2 - 25\lambda + 108$$

Then $p(A) = 2A^2 - 25A + 108I = \begin{bmatrix} 85 & 4 & -30 \\ 0 & 51 & 0 \\ 0 & -11 & 40 \end{bmatrix}$

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Answer of Question 5

(a)The matrix form AX = B, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$

If $G = [A \; \vdots \; B]$

This system has three types of solutions :

(i)One solution if rank A = rank G = 3.

(ii)Infinite number of solutions if rank A = rank G < 3.

(iii)No solution if rank A ≠ rank G.

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(b) $G = \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & 1 & -1 & 4 \\ 3 & 0 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 5 & -5 & 0 \\ 0 & 6 & -6 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & -4 \end{array}\right].$

It has no solution.

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Final Exam and ILOs

Course Title: Linear Algebra Code: EMM 402

Questions	ILOs				
	Knowledge and Understanding		Intellectual Skills		Professional and Practical Skills
	2.1.1	2.1.2	2.2.3	2.2.7	2.3.2
Q1	√		√		
Q2			√	√	
Q3	√		√		
Q4		√			√
Q5	√	√		√	√

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