Benha University
Faculty of Engineering (Shoubra)
Engineering Mathematics and
Physics Department

Ordinary and partial differential equations
Code: EMM 401 January 2016
Time allowed: 3 hours
Scores :200 marks

1-i) Discuss two different numerical methods for solving the following system of ordinary differential equation $y`` = f(t, y`, y, x)$ , $x`` = g(t, x`, y, x)$ given $y(t_0) = a$ , $y`(t_0) = b$ and $x(t_0) = c$, $x`(t_0) = d$. **[30]**

1-ii) Use modified Euler method to find $y(t_2)$ with a given h. **[20]**

1-iii) Apply the above methods in solving the following differential equation **[50]**

$$x` - 3y` = -2t + x - 2y - 7,\ 2x` + y` = 10t + y + 3 - t^2,\ x(1) = 2,\ y(1) = 3$$

2-i) Discuss the solution U(x,y) of the following P.D.E. analytically: **[30]**

a- $U_{tt} = c^2 U_{xx}$, $0 < x < L$, with B.C.: $U(0,t) = U(L,t) = 0$ and I.C. : $U(x,0) = f(x)$, $U_t(x,0) = g(x)$

b - $U_t = c\, U_{xx}$, $0 < x < L$, with B.C.: $U(0,t) = U(L,t) = 0$ and I.C. : $U(x,0) = f(x)$.

2-ii) Represent above P.D.E in square mesh using finite difference method taking h = k. **[20]**

2-iii) If $L = 1$, $f(x) = x(x-1)$, $g(x) = 0$, $c = 1$, solve the above differential equations numerically & analytically. **[50]**

Modified Euler method: $y_{i+1} = y_i + (h/2)[f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))]$

Good luck **Board of examiners Dr. eng Khaled El Naggar**

Model answer

Answer of question 1

1-i) **Using Taylor**

Let y\` = z , z\` = f(t, z, y, x) and let x\` = w, w\` = g(t, w, y, x), thus y\`\` = z\` = f(t, z, y, x) and y\`\`\`= z\`\`= h(t, z, y, x, z\`, y\`, x\`). Also x\`\`= w\`= g(t, w, y, x) & x\`\`\` = w\`\`= s(t, w, y, x, w\`, y\`, x\`).

$$y(t) = y_0 + \frac{t - t_0}{1!} y_0^{(1)} + \frac{(t - t_0)^2}{2!} y_0^{(2)} + \frac{(t - t_0)^3}{3!} y_0^{(3)} + \cdots ,$$

$$z(t) = z_0 + \frac{t - t_0}{1!} z_0^{(1)} + \frac{(t - t_0)^2}{2!} z_0^{(2)} + \frac{(t - t_0)^3}{3!} z_0^{(3)} + \cdots ,$$

$$x(t) = x_0 + \frac{t - t_0}{1!} x_0^{(1)} + \frac{(t - t_0)^2}{2!} x_0^{(2)} + \frac{(t - t_0)^3}{3!} x_0^{(3)} + \cdots ,$$

$$w(t) = w_0 + \frac{t - t_0}{1!} w_0^{(1)} + \frac{(t - t_0)^2}{2!} w_0^{(2)} + \frac{(t - t_0)^3}{3!} w_0^{(3)} + \cdots ,$$

$x_0 = c$, $y_0 = a$, $z_0 = b$, $w_0 = d$, $y_0^{(1)} = z_0 = b$, $y_0^{(2)} = z_0^{(1)} = f(t_0, b, a, c) = f_0$, $y_0^{(3)} = z_0^{(2)} = h(t_0, b, a, c, f_0, d)$ $x_0^{(1)} = w_0 = d$, $x_0^{(2)} = w_0^{(1)} = g(t_0, d, a, c) = g_0$, $x_0^{(3)} = w_0^{(2)} = s(t_0, d, a, c, g_0, b)$

Since f(t, z, y, x) and g(t, w, y, x) are given functions and t_0, a, b, c, d are given constants, hence we can get y(t) and x(t)

Using Picard

$y_{n+1} = y_0 + \int_{t_0}^{t} z_n dt$, therefore $y_1 = y_0 + \int_{t_0}^{t} z_0 dt = a + \int_{t_0}^{t} b\, dt = a + b(t - t_0)$ and $y_2 = y_0 + \int_{t_0}^{t} z_1 dt$,

where

$z_{n+1} = z_0 + \int_{t_0}^{t} f(t, z_n, y_n, x_n) dt$, hence $z_1 = z_0 + \int_{t_0}^{t} f(t, z_0, y_0, x_0) dt = b + \int_{t_0}^{t} f(t, b, a, c) dt$.

Therefore $y_2 = y_0 + \int_{t_0}^{t} [b + \int_{t_0}^{t} f(t, b, a, c) dt]\, dt$.

$x_{n+1} = x_0 + \int_{t_0}^{t} w_n dt$, therefore $x_1 = x_0 + \int_{t_0}^{t} w_0 dt = c + \int_{t_0}^{t} d\, dt = c + d(t - t_0)$ and $x_2 = x_0 + \int_{t_0}^{t} w_1 dt$,

where

$w_{n+1} = w_0 + \int_{t_0}^{t} g(t, w_n, y_n, x_n) dt$, hence $w_1 = w_0 + \int_{t_0}^{t} g(t, w_0, y_0, x_0) dt = d +$

$\int_{t_0}^{t} g(t, d, a, c) dt$.

Therefore $x_2 = x_0 + \int_{t_0}^{t} [d + \int_{t_0}^{t} f(t, d, a, c) dt]\, dt$.

1-ii) **Use Modified Euler method:**

$y_{i+1} = y_i + (h/2)\, [2z_i + h\, f(t_i, z_i, y_i, x_i)]$, therefore $y_1 = y_0 + (h/2)\, [2z_0 + h\, f(t_0, z_0, y_0, x_0)] \Rightarrow$

$y_1 = a + (h/2)\, [2b + h\, f(t_0, b, a, c)]$ and $y_2 = y_1 + (h/2)\, [2z_1 + h\, f(t_1, z_1, y_1, x_1)]$, where

$x_1 = x_0 + (h/2)\, [2w_0 + h\, g(t_0, w_0, y_0, x_0)] \Rightarrow x_1 = x_0 + (h/2)\, [2d + h\, g(t_0, d, a, c)]$

1-iii) $x` = x^2 - 2tx + y - 2t = f(x,y,t)$, $y` = y - x^2 - 2tx + 2t + 3 = \varphi(x,y,t)$, $x_0 = 2$, $y_0 = 3$, $t_0 = 1$

$$y_{i+1} = y_i + (h/2)[\varphi(t_i, x_i, y_i) + \varphi(t_{i+1}, x_i + hf(t_i, x_i, y_i), y_i + h\varphi(t_i, x_i, y_i))]$$

Put i = 0, therefore

$$y_1 = y_0 + (h/2)[\varphi(t_0, x_0, y_0) + \varphi(t_1, x_0 + hf(t_0, x_0, y_0), y_0 + h\varphi(t_0, x_0, y_0))] = 2.9585$$

By picard

$$y_{n+1} = y_0 + \int_{t_0}^{t} (x_n z_n + 28x_n - y_n)\, dt, \quad x_{n+1} = x_0 + \int_{t_0}^{t} -10(x_n - y_n)\, dt, \quad z_{n+1} = z_0 + \int_{t_0}^{t} (x_n y_n - 8z_n/3)\, dt,$$

$y_0 = -1$, $x_0 = 2$, $t_0 = 0$, $z_0 = 3$, thus $x_1 = x_0 + \int_{t_0}^{t} -10(x_0 - y_0)\, dt$, $y_1 = y_0 + \int_{t_0}^{t} (x_0 z_0 + 28x_0 - y_0)\, dt$ and $z_1 = z_0 + \int_{t_0}^{t} (x_0 y_0 - 8z_0/3)\, dt$, therefore $x_1 = 2-30t$, $y_1 = -1+51t$, $z_1 = 3-10t$. Similarly, $x_2 = x_0 + \int_{t_0}^{t} -10(x_1 - y_1)\, dt$, $y_2 = y_0 + \int_{t_0}^{t} (x_1 z_1 + 28x_1 - y_1)\, dt$ and $z_2 = z_0 + \int_{t_0}^{t} (x_1 y_1 - 8z_1/3)\, dt$, therefore $x_2 = 2 - 30t + 405t^2$, $y_2 = -1 + 51t - (781/2)t^2 - 100t^3$, $z_2 = 3 - 10t + (238/3)t^2 - 510t^3$.

2nd : <u>using Euler,</u> $x_{n+1} = x_n + h[-10(x_n - y_n)]$, $y_{n+1} = y_n + h[-x_n z_n + 28x_n - y_n]$, thus $x_1 = x_0 + h[-10(x_0 - y_0)] = 0.5 = x(0.05)$, $y_1 = y_0 + h[-x_0 z_0 + 28x_0 - y_0] = 1.55 = y(0.05)$, therefore $x(0.1) = x_2 = x_1 + h[-10(x_1 - y_1)] = 1.025$

Answer of question 2

2-i)

a) We use Separation method to solve the Wave equation, so that the solution is expressed as $U(x,t) = \phi(x)\Psi(t)$, therefore $U_{xx} = \phi''(x)\,\Psi(t)$ and $U_{tt} = \phi(x)\,\Psi''(t)$, thus $c^2\phi''(x)\,\Psi(t) = \phi(x)\,\Psi''(t)$.

Therefore $\frac{\phi''(x)}{\phi(x)} = \frac{1}{c^2}\frac{\psi''(x)}{\psi(x)} = -\lambda$, where λ is positive constant.

Thus $\phi''(x)+\lambda\,\phi(x)=0$, the characteristic equation is $m^2+\lambda=0$, so

$\phi(x)= c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$.

And $\Psi''(t)+c^2\lambda\,\Psi(t)=0$, the characteristic equation is $n^2+c^2\lambda=0$, so

$\Psi(t)= c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t$.

Therefore $U(x,t) = (c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x)(c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t)$.

But $U(0,t) = 0$, therefore $c_1\,(c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t) = 0$, thus $c_1 = 0$, hence $U(x,t) = (c_2 \sin\sqrt{\lambda}\,x)(c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t)$.

Since $U(L,t) = 0$, therefore $(c_2 \sin\sqrt{\lambda}\,L)(c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t) = 0$, but $c_2 \neq 0$, thus $\sin\sqrt{\lambda}\,L = 0$, hence $\sqrt{\lambda}\,L = n\pi \Rightarrow \lambda = (\frac{n\pi}{L})^2$, n = 1, 2, 3, …. Therefore $\phi(x) = (c_2 \sin(\frac{n\pi}{L})\,x)$, thus $U(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L})x[\,A_n \cos(\frac{cn\pi}{L})\,t + B_n \sin(\frac{cn\pi}{L})t]$

But $U(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L})x$, which is Fourier sine series such that

$$A_n = \frac{2}{L}\int_0^L f(x)\,\sin(\frac{n\pi}{L})x\,dx$$

Since $U_t(x,t) = \sum_{n=1}^{\infty} (\frac{cn\pi}{L})\sin(\frac{n\pi}{L})x[\,-A_n \sin(\frac{cn\pi}{L})t + B_n \cos(\frac{cn\pi}{L})t]$

And $U_t(x,0) = g(x)$, therefore $\sum_{n=1}^{\infty} B_n (\frac{cn\pi}{L})\sin(\frac{n\pi}{L})x = g(x)$, which is Fourier sine series such that $B_n(\frac{cn\pi}{L}) = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi}{L})x\,dx$, therefore

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin(\frac{n\pi}{L})x\,dx$$

b) We use Separation method to solve the Heat equation, so that the solution is expressed as $U(x,t) = \phi(x)\Psi(t)$, therefore $U_{xx} = \phi''(x)\,\Psi(t)$ and $U_t = \phi(x)\,\Psi'(t)$, thus $c\phi''(x)\,\Psi(t) = \phi(x)\,\Psi'(t)$.

Therefore $\frac{\phi''(x)}{\phi(x)} = \frac{1}{c}\frac{\psi'(x)}{\psi(x)} = -\lambda$, where λ is positive constant.

Thus $\phi''(x) + \lambda\,\phi(x) = 0$, the characteristic equation is $m^2 + \lambda = 0$, so

$\phi(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$ and $\Psi'(t) + c\lambda\,\Psi(t) = 0$, the characteristic equation is $n + c\lambda = 0$, so $\Psi(t) = c_3\,e^{-c\lambda\,t}$

Therefore $U(x,t) = (c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x)(c_3\,e^{-c\lambda\,t})$.

But $U(0,t) = 0$, therefore $U(0,t) = (c_1)(c_3\,e^{-c\lambda\,t}) = 0$, thus $c_1=0$, hence $U(x,t) = (c_2 \sin\sqrt{\lambda}\,x)(c_3\,e^{-c\lambda\,t})$, and $U(L,t) = 0$, therefore: $(c_2 \sin\sqrt{\lambda}\,L)(c_3\,e^{-c\lambda\,t}) = 0$ and $c_2 \neq 0$, thus $\sin\sqrt{\lambda}\,L = 0$, hence $\sqrt{\lambda}\,L = n\pi \Rightarrow \lambda = (\frac{n\pi}{L})^2$, $n = 1, 2, 3, \ldots$. Therefore $\phi(x) = (c_2 \sin(\frac{n\pi}{L})x)$, thus $U(x,t) = \sum_{n=1}^{\infty} A_n e^{(-\frac{cn\pi}{L})\,t}\sin(\frac{n\pi}{L})x$, but $U(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L})x$, which is Fourier sine series such that $A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L})x\,dx$

2-ii) To solve the above equation numerically , we use graphical representation of partial equations such that: $u_x = \frac{u_{i+1,j} - u_{i,j}}{h} = \frac{u_{i,j} - u_{i-1,j}}{h}$, $u_y = \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i,j} - u_{i,j-1}}{k}$,

$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$, $u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$, $u_{xy} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4hk}$, but $U_{tt} = c^2\ U_{xx}$, therefore $\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = c^2\ \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$ and for $U_t = c\ U_{xx}$, therefore $\frac{u_{i,j+1} - u_{i,j}}{k} = c\ \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$

But h = k , therefore $u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = c^2\ [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] \Rightarrow$

$u_{i,j+1} = 2(1-c^2)\ u_{i,j} + c^2\ [u_{i+1,j} + u_{i-1,j}] - u_{i,j-1}$

and for $U_t = c\ U_{xx}$, the formula is $h[u_{i,j+1} - u_{i,j}] = c\ [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$

2-iii-a) $U(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L})x[\ A_n \cos(\frac{cn\pi}{L})\ t + B_n \sin(\frac{cn\pi}{L})t]$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L})x\ dx,\ B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin(\frac{n\pi}{L})x\ dx,$$

2-iii-b) $U(x,t) = \sum_{n=1}^{\infty} A_n e^{(-\frac{cn\pi}{L})\ t} \sin(\frac{n\pi}{L})x, A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L})x\ dx$

L = 1, c = 1 , g(x) = 0, f(x) = x(x-1)

1. **Overall aims of course**

By the end of the course the students will be able to:

- Solve ordinary and partial differential equations numerically
- Recognize finite difference method in solving P.D.E.
- Describe error analysis and stability for P.D.E.

2. **Intended Learning outcomes of Course (ILOs)**

a. **Knowledge and Understanding:**

2.1.1 Identify theories, fundamentals of ordinary and partial differential equations [**Q1, Q2**]

2.1.3 Recognize the developments of finite difference method in solving P.D.E. [**Q2**]

2.1.4 Summarize the moral and legal principles of error analysis and stability [**Q1**]

b. **Intellectual Skills**

2.2.5 Assess solutions of partial differential equations using finite difference method. [**Q2**]

c. **Professional and Practical Skills**

2.3.1 Interpret professional skills in estimating error analysis and stability. [**Q1**]

d. **General and Transferable Skills**

2.4.1 Communicate effectively using researches of new topics about solutions of ordinary and partial differential equations .

2.4.5 Assess the performance of error analysis and stability

2.4.6 Work in a group and manage time effectively