Benha University
Faculty of Engineering-Shoubra
Electrical Engineering Department
1st Year Communications

Final Term Exam
Date: 11th of Jan 2017
Mathematics 2A
Duration: 3 hours

- Answer all the following questions
- Illustrate your answers with sketches when necessary
- The exam consists of one page
- No. of questions: 4
- Total mark: 80 marks

Question 1 [15 marks]

Test the following series:

(a) $\sum_{k=0}^{\infty} \frac{2^k+3^k}{4^k+5^k}$ (b) $\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln(k)}$

Question 2 [25 marks]

Solve the following ODEs:

(a) $(x+y+4)dx + (2x+2y-1)dy = 0$ (b) $y'' - 6y' + 9y = x^{-2}e^{3x}$

(c) $y'' + 4y = x\sin(2x)$

Question 3 [25 marks]

(a) Evaluate $\iint_S (\vec{\nabla} \times \vec{F}).\vec{n}\, dS$ where $\vec{F} = (x^2+y-4)\vec{i} + (3xy)\vec{j} + (2xz+z^2)\vec{k}$ and S is the surface bounded by the parabolid $z = 4-(x^2+y^2)$, $z \geq 0$.

(b) Verify Green's theorem for the integral $\oint_C xy\,dx + (x+y)\,dy$ where C is the path given by the two curves $x = y^2$, $y = x^2$.

Question 4 [15 marks]

(a) If $u = \frac{x+y+z}{x-y+2z} + \sin^{-1}(\frac{x}{y}) + 4$, show that $xu_x + yu_y + zu_z = 0$.

(b) Discuss the maxima and minima of $f(x,y) = \frac{1}{3}x^3 + \frac{4}{3}y^3 - x^2 - 3x - 4y - 3$

Dr. Ibrahim Sakr Good Luck Dr. Eng. Tarek Sallam

Model Answer

Answer of Question 1

(a) Take $b_k = \frac{2^k+3^k}{5^k} = (\frac{2}{5})^k + (\frac{3}{5})^k$ which is the sum of two conv. geometric series (r < 1) and the resultant series is conv., and since $a_k < b_k$, then the given series is conv.

(b) $|a_k| = \frac{1}{k \ln k}$, and since $|a_k| < |a_{k+1}|$ and $\lim\limits_{k\to\infty} |a_k| = 0$, then the series is conv. And since $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$ is div. (using integral test), then the given series is conditionally conv.

Answer of Question 2

(a) $\frac{dy}{dx} = y' = \frac{x+y+4}{-(2x+2y-1)}$ which are two parallel lines, then put u = x + y, so $u' = 1 + y' \rightarrow y' = u' - 1 \rightarrow \frac{du}{dx} - 1 = \frac{u+4}{-(2u-1)} \rightarrow \frac{du}{dx} = \frac{5-u}{1-2u} \rightarrow \frac{1-2u}{5-u} du = dx \rightarrow \left(2 - \frac{9}{5-u}\right) du = dx \rightarrow 2u + 9\ln(5-u) = x + c \rightarrow 2y + x + 9\ln(5 - x - y) = c$

(b) The aux. eq. is $m^2 - 6m + 9 = 0$, its roots are 3, 3, then $y_c = (c_1 + c_2 x)e^{3x}$. Then $u = e^{3x}$, $v = xe^{3x}$, $w = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$

$A = -\int \frac{xe^{3x}.x^{-2}e^{3x}}{e^{6x}} dx = -\ln x + c_1, B = \int \frac{e^{3x}.x^{-2}e^{3x}}{e^{6x}} dx = -\frac{1}{x} + c_2$

$\therefore y = (-\ln x + c_1)\ e^{3x} + (-\frac{1}{x} + c_2)\ xe^{3x}$

$\therefore y = (c_1 + c_2 x)e^{3x} + (-\ln x - 1)e^{3x}$

(c) The aux. eq. is $m^2 + 4 = 0$, its roots are 2i, -2i, then $y_c = c_1 \cos 2x + c_2 \sin 2x$. Then $y_p = \frac{1}{D^2+4} \mathrm{Im}\left(xe^{2ix}\right) = \frac{1}{64}[4x \sin 2x - (1 + 8x^2)\cos 2x]$

$\therefore y = y_c + y_p$

Answer of Question 3

(a) Apply the Stoke's theorem: $\oint_C \vec{F}\,.\,\overrightarrow{dr} = \iint_S (\vec{\nabla} \times \vec{F})\,.\,\vec{n}\, dS$, where C is the circle on xy-plane $x^2 + y^2 = 4$, z = 0, then $\oint_C \vec{F}\,.\,\overrightarrow{dr} = \oint_C (x^2 + y - 4)dx + (3xy)dy$, changing to polar coordinates: x = 2cosθ, dx = -2sinθ, y = 2sinθ, dy = 2cosθ

$$\therefore I_l = \int_0^{2\pi} (16(\cos\theta)^2 \sin\theta - 4(\sin\theta)^2 + 8\sin\theta) d\theta = -2\theta|_0^{2\pi} = -4\pi$$

(b) Green's th.: $\oint_C M\,dx + N\,dy = \iint_S \left(N_x - M_y\right) dxdy$

I_1 along $y = x^2$: dy = 2xdx, so $I_1 = \int_0^1 (2x^2 + 3x^3) dx = \frac{17}{12}$

I_2 along $x = y^2$: dx = 2ydy, so $I_2 = \int_1^0 (y + y^2 + 2y^4) dx = -\frac{37}{30} \rightarrow I_l = I_1 + I_2 = \frac{11}{60}$

$$M = xy, N = x + y \to I_s = \int_0^1 \left[\int_{x=y^2}^{\sqrt{y}} (1-x)dx\right] dy = \int_0^1 (\sqrt{y} - \frac{y}{2} - y^2 + \frac{y^4}{2})dy$$

$$= \frac{11}{60} \to I_l = I_s$$

Answer of Question 4

(a) Since u is hom. of degree 0, then $xu_x + yu_y + zu_z = 0$

(b) $f_x = x^2 - 2x - 3 = 0 \rightarrow x = 3, -1$, and $f_y = 4y^2 - 4 = 0 \rightarrow y = \pm 1 \rightarrow$ the critical points are (3, 1), (3, -1), (-1, 1), (-1, -1), $f_{xx} = 2x - 2 > 0$, $f_{yy} = 8y$, $f_{xy} = 0$,

$\Delta = f_{xx}f_{yy} - (f_{xy})^2 = 16y(x-1)$:

at (3, 1): $\Delta = 32 > 0, f_{xx} = 4 > 0 \to$ min $\to f(3,1) = \frac{-44}{3}$

at (3, -1): $\Delta = -32 < 0 \to$ saddle point

at (-1, 1): $\Delta = -32 < 0 \to$ saddle point

at (-1, -1): $\Delta = 32 > 0, f_{xx} = -4 < 0 \to$ max $\to f(-1,-1) = \frac{4}{3}$