Benha University
Faculty of Engineering- Shoubra
Mechanical Engineering Department
1st Year Mechanical (production) تخلفات

Final Term Exam
Date: 24 – 12 - 2016
Mathematics 1- Code : EMP191
Duration : 3 hours

- Answer all the following questions
- The exam. Consists of one page
- No. of questions:3
- Total marks: 105

1-a) Test the following series for convergence: **[15]**

i) $\sum_{n=1}^{\infty}\frac{5^n+7^n}{3^n+2^n}$ ii) $\sum_{n=1}^{\infty}[\frac{2n^2+1}{n^2+1}]^n$ iii) $\sum_{n=1}^{\infty}\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$

1-b) Solve the following differential equations: **[30]**

i) $y` = \frac{\cos y - ye^x}{e^x + x\sin y}$ ii) $y` = (y/x) + \tan(y/x)$ iii) $y` = (y/2x) - (xy)^3$ iv) $y'' + 9y = \operatorname{cosec} 3x$

v) $y'' + 4y' + 3y = \cosh 2x \sin 3x \cos 2x$ vi) $2y'' + 5y' + 2y = x^2 \sinh 4x$

2-a) Evaluate the following integrals **[15]**

i) $\int_c (xy + \ln x)\, dy$, c is the arc of parabola $y = x^2$ from (1,1) to (3,9)

ii) $\int_c 4x^3\, ds$, c is the line segment from (-2,-1) to (1,2)

iii) $\iint_D \frac{1}{x}\sin(\frac{y}{x})\, dxdy$, where D: area bounded by $y = x^2$ and x axis from $x = \pi/2$ to $x = \pi$

2-b) Find interval of convergence **[15]**

i) $\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!\,x^{2n+1}}$ ii) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1} n!\, x^{2n}}{n^2+3}$ iii) $\sum_{n=1}^{\infty}\frac{(-1)^n (x-6)^n}{n\, 3^n}$

3-a) Find envelope of the function $(x-\cos\alpha)^2 + (y-\sin\alpha)^2 = \rho$ **[15]**

3-b) Expand $\sin^{-1}(\frac{x+y}{x-y})$ using Taylor series about $(1, \pi/2)$ & find $xf_x + yf_y$ **[15]**

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Model answer

1a-i) By ratio test, we get that $\lim_{n\to\infty} (\frac{5^{n+1}+7^{n+1}}{3^{n+1}+2^{n+1}})(\frac{3^n+2^n}{5^n+7^n}) = \lim_{n\to\infty} \frac{7^{n+1}}{3^{n+1}}[\frac{(5/7)^{n+1}+1}{1+(2/3)^{n+1}}]$ $\frac{3^n}{7^n}[\frac{3^n+2^n}{5^n+7^n}]$ = 7/3 > 1, therefore the series is divergent.

1a-ii) $\lim_{n\to\infty}\sqrt[n]{[\frac{2n^2+1}{n^2+1}]^n} = \lim_{n\to\infty}[\frac{2n^2+1}{n^2+1}] = 2 > 1$, thus $\sum_{n=1}^{\infty}[\frac{2n^2+1}{n^2+1}]^n$ is divergent.

1a-iii) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n^{-1/3}$ is divergent, therefore $\sum_{n=1}^{\infty}\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ is divergent.

1b-i) $(\cos y - ye^x)dx - (e^x + x\sin y)dy = 0 \Rightarrow$ $M_y = N_x = -\sin y - e^x \Rightarrow$ therefore the D.E. is exact, thus

$f_x = \cos y - y\,e^x \Rightarrow f(x, y) = x\cos y - y\,e^x + g(y) \Rightarrow f_y = -x\sin y - e^x + g`(y) = N(x,y) \Rightarrow g`(y) = 0$, therefore $g(y) = c \Rightarrow f(x, y) = x\cos y - y\,e^x + c$

1b-ii) Put $y = vx \Rightarrow dy = vdx + xdv \Rightarrow vdx + xdv = (v + \tan v)dx \Rightarrow \cot v\,dv = dx/x \Rightarrow \ln\sin(y/x) = \ln x$

1b-iii) $y` = (y/2x) - (xy)^3$ is Bernoulli D.E., thus $y^{-3}y` - y^{-2}/2x = -x^3$. Put $z = y^{-2} \Rightarrow z` = -2y^{-3}y` \Rightarrow$ $z` + z/x = 2x^3$ which is linear D.E. whose solution is $zx = -2x^5/5 + c$, so $xy^{-2} = -2x^5/5 + c$ is the solution of D.E.

1b-iv) $y`` + 9y = \text{cosec}(x)$ has homogeneous and particular solution so that the characteristic equation is $m^2 + 9 = 0 \Rightarrow m = -3i, 3i$, thus $y_H = (c_1\cos 3x + c_2\sin 3x)$ and so the particular solution is: $y_P = u_1(x)\cos 3x + u_2(x)\sin 3x$, and $y_1(x) = \cos 3x$, $y_2(x) = \sin 3x$ where $u_1(x) = -\int\frac{y_2 g(x)}{W(y_1, y_2)}dx$, $u_2(x) = \int\frac{y_1 g(x)}{W(y_1, y_2)}dx$, where $W(y_1, y_2) = y_2' y_1 - y_2 y_1' = 3$, g(x) = secx, thus $u_1(x) = -\int\frac{\sin 3x\ \text{cosec} 3x}{1}dx = -x$ & $u_2(x) = \int\frac{\cos 3x\ \text{cosec} 3x}{1}dx = \ln\sin x$

1b-v) $m^2 + 4m + 3 = 0 \Rightarrow (m+1)(m+3) = 0 \Rightarrow m = -1, -3 \Rightarrow y_H = c_1 e^{-x} + c_2 e^{-3x}$ and

$$y_P = \frac{1}{2(D^2+4D+3)}\cosh 2x[\sin x + \sin 5x] = \frac{1}{4(D^2+4D+3)}[e^{2x} - e^{-2x}][\sin x + \sin 5x]$$

$$y_P = \frac{1}{4(D^2+4D+3)}e^{2x}[\sin x + \sin 5x] - \frac{1}{4(D^2+4D+3)}e^{-2x}[\sin x + \sin 5x]$$

$$y_P = e^{2x}\frac{1}{4((D+2)^2+4(D+2)+3)}[\sin x + \sin 5x] - e^{-2x}\frac{1}{4(D-2)^2+4(D-2)+3)}[\sin x + \sin 5x]$$

$$y_P = e^{2x}\frac{1}{4(D^2+8D+15)}[\sin x + \sin 5x] - e^{-2x}\frac{1}{4(D^2-1)}[\sin x + \sin 5x]$$

$$y_p = e^{2x}\frac{1}{4(8D+14)}\sin x + e^{2x}\frac{1}{4(8D-10)}\sin 5x + \frac{e^{-2x}}{8}\sin x + \frac{e^{-2x}}{104}\sin 5x$$

$$y_p = e^{2x}\frac{4D+7}{8(16D^2-49)}\sin x + e^{2x}\frac{4D+5}{8(16D^2-25)}\sin 5x + \frac{e^{-2x}}{8}\sin x + \frac{e^{-2x}}{104}\sin 5x$$

$$y_p = -e^{2x}\frac{4\cos x + 7\sin x}{520} - e^{2x}\frac{20\cos 5x + 5\sin 5x}{3400} + \frac{e^{-2x}}{8}\sin x + \frac{e^{-2x}}{104}\sin 5x$$

1b-vi) $2m^2 + 5m + 2=0 \Rightarrow (2m + 1)(m + 2) = 0 \Rightarrow m = -1/2, -2 \Rightarrow y_H = c_1 e^{-(1/2)x} + c_2 e^{-2x}$ and

$$y_p = \frac{1}{2D^2+5D+2}x^2\sinh 4x = \frac{1}{2(2D^2+5D+2)}x^2[e^{4x}-e^{-4x}]$$

$$y_p = \frac{1}{2(2D^2+5D+2)}x^2e^{4x} - \frac{1}{2(2D^2+5D+2)}x^2e^{-4x}$$

$$y_p = e^{4x}\frac{1}{2(2(D+4)^2+5(D+4)+2)}x^2 - e^{-4x}\frac{1}{2(2(D-4)^2+5(D-4)+2)}x^2$$

$$y_p = e^{4x}\frac{1}{2(2D^2+21D+54)}x^2 - e^{-4x}\frac{1}{2(2D^2-11D+14)}x^2$$

$$y_p = e^{4x}\frac{1}{108(\frac{2D^2+21D}{54}+1)}x^2 - e^{-4x}\frac{1}{28(\frac{2D^2-11D}{14}+1)}x^2$$

$$y_p = \frac{e^{4x}}{108}(1+\frac{2D^2+21D}{54})^{-1}x^2 - \frac{e^{-4x}}{28}(1+\frac{2D^2-11D}{14})^{-1}x^2$$

$$y_p = \frac{e^{4x}}{108}(1-(\frac{2D^2+21D}{54})+(\frac{2D^2+21D}{54})^2)x^2 - \frac{e^{-4x}}{28}(1-(\frac{2D^2-11D}{14})+(\frac{2D^2-11D}{14})^2)x^2$$

$$y_p = \frac{e^{4x}}{108}(1-\frac{D^2}{27}-\frac{7D}{18}+\frac{49D^2}{324})x^2 - \frac{e^{-4x}}{28}(1-\frac{D^2}{7}+\frac{11D}{14}+\frac{121D^2}{196})x^2$$

$$y_p = \frac{e^{4x}}{108}(1-\frac{7D}{18}+\frac{37D^2}{324})x^2 - \frac{e^{-4x}}{28}(1+\frac{11D}{14}+\frac{93D^2}{196})x^2$$

$$y_p = \frac{e^{4x}}{108}(x^2-\frac{7x}{9}+\frac{37}{162}) - \frac{e^{-4x}}{28}(x^2+\frac{11x}{7}+\frac{93D^2}{98})$$

2a-i) $y = x^2 \Rightarrow dy = 2xdx \Rightarrow \int_c (xy + \ln x)\,dy = \int_1^3 [x(x^2)+\ln x](2x)dx = 2\int_1^3 [x^4 + x\ln x]dx$

$$= (2(\frac{x^5}{5}) + 2[\frac{x^2}{2}\ln x - \frac{x^2}{4}])_1^3$$

2a-ii) Since $ds = \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt$ and the contour is $y = x+1 \Rightarrow$ if $x = t$, therefore $y = t+1 \Rightarrow$

$ds = \sqrt{(1)^2+(1)^2}\,dt = \sqrt{2}dt \Rightarrow \int_c 4x^3\,ds = \int_{-2}^{1} 4\sqrt{2}t^3dt = \sqrt{2}\,t^4\Big|_{-2}^{1} = \sqrt{2}[1-16] = -15\sqrt{2}$

2a-iii) $\iint_D \frac{1}{x}\sin(\frac{y}{x})\,dxdy = \int_{x=\pi/2}^{x=\pi}[\int_{y=0}^{y=x^2}\frac{1}{x}\sin(\frac{y}{x})\,dy]dx = -\int_{x=\pi/2}^{x=\pi}\cos(\frac{y}{x})\Big|_{y=0}^{y=x^2}\,dx$

$= -\int_{x=\pi/2}^{x=\pi}\cos(x)dx = -\sin x\Big|_{\pi/2}^{\pi} = 1$

2b-i) Since $U_n = \frac{(-1)^n}{(2n+1)!\,x^{2n+1}}$, and $U_{n+1} = \frac{(-1)^{n+1}}{(2n+3)!\,x^{2n+3}}$, hence the ratio $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(-1)^{n+1}(2n+1)!\,x^{2n+1}}{(-1)^n(2n+3)!\,x^{2n+3}}\right| = \left|-\frac{1}{(2n+3)(2n+2)x^2}\right|$,thus

$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|-\frac{1}{(2n+3)(2n+2)x^2}\right| = 0 < 1 = 0$ hence series is convergent for all x.

2b-ii) Since $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(-1)^n(n+1)!x^{2n+2}[n^2+3]}{(-1)^{n-1}n!\,x^{2n}[(n+1)^2+3]}\right| = \left|-\frac{(n+1)x^2[n^2+3]}{[(n+1)^2+3]}\right|$, thus

$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|-\frac{(n+1)x^2[n^2+3]}{[(n+1)^2+3]}\right| = \infty > 1 = 0$ hence series is divergent for all x.

2b-iii) Since $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(-1)^{n+1}(x-6)^{n+1}[n\,3^n]}{(-1)^n(x-6)^n[(n+1)\,3^{n+1}]}\right| = \left|-\frac{(x-6)^{n+1}[n\,3^n]}{(x-6)^n[(n+1)\,3^{n+1}]}\right|$ thus

$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(x-6)^{n+1}[n\,3^n]}{(x-6)^n[(n+1)\,3^{n+1}]}\right| = \left|\frac{x-6}{3}\right| < 1 \Rightarrow |x-6| < 3 \Rightarrow 3 < x < 9$

3-a) $\frac{\partial}{\partial\alpha}[(x-\cos\alpha)^2 + (y-\sin\alpha)^2 = \rho] \Rightarrow 2(x-\cos\alpha)\sin\alpha = 2(y-\sin\alpha)\cos\alpha \Rightarrow \tan\alpha = y/x$, therefore $-x\sin\alpha + y\cos\alpha = 0$, thus $\tan\alpha = y/x$, so $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$, hence envelope is $\sqrt{x^2+y^2} - 1 = \pm\rho$.