Benha University
Faculty of Engineering- Shoubra
Mechanical Engineering Department
1st Year Electrical (power) تخلفات

Final Term Exam
Date:December 2016
Mathematics 2A-Code: EMP171
Duration : 3 hours

- Answer all the following questions
- Illustrate your answers with sketches when necessary.
- The exam. Consists of one page
- No. of questions:3
- Total Mark: 100 Marks

1-a) Test the following series for convergence: **[15]**

i) $\sum_{n=1}^{\infty}\left[\frac{5n^2-3n^3}{7n^3+2}\right]^{5n}$ ii) $\sum_{n=1}^{\infty}\frac{n^2}{(3n+1)!}$ iii) $\sum_{n=1}^{\infty}\frac{n^2}{\sqrt[5]{n^2+1}}$

1-b) Solve the following differential equations: **[15]**

i) $x\sec^2 y\,dx = e^{-x}dy$ ii) $\frac{dy}{dx}=\frac{x+y+3}{x-y+1}$ iii) $y' - y\tan x = y^2\cos^3 x$

1-c) Find Envelope of $f(x, y, \alpha) = x\cos\alpha + y\sin\alpha = P$, α is the parameter **[5]**

2-a) Find the interval of convergence for the following series: **[10]**

i) $\sum_{n=1}^{\infty}\frac{3^n}{(n^2+1)(x-2)^n}$ ii) $\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^{2n}}{2n!}$

2-b) Find the dimensions of the rectangular box with the largest volume with faces parallel to the coordinate planes that can be inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$. **[10]**

2-c) Solve the following differential equations: **[15]**

i) $(y + \ln(x))dx+(x+y^2)\,dy=0$ ii) $y''+y = 1+\tan x$ iii) $y''+2y'+2y=e^x\sin^2(2x)$

3-a) Expand the function $f(x, y) = \ln\left(\frac{x+y}{x-y}\right)$ using Taylor series about (0,1) **[10]**

3-b) Evaluate the following integrals **[10]**

i) $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\sqrt{x^2+y^2}\,dydx$ ii) $\int_c \sin(\pi y)\,dx + yx^2 dy$, c is line from (0,2) to (1,4)

3-c) Find the volume of the parallelepiped spanned by the vectors **[10]**

$u = (1,0,2) \quad v = (0,2,3) \quad w = (0,1,3)$

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Model answer

1a-i) $\lim_{n\to\infty} \sqrt[n]{[\frac{5n^2-3n^3}{7n^3+2}]^{5n}} = \lim_{n\to\infty}[\frac{5n^2-3n^3}{7n^3+2}]^5 = [-\frac{3}{7}]^5 < 1$, thus $\sum_{n=1}^{\infty}[\frac{5n^2-3n^3}{7n^3+2}]^{5n}$ is convergent.

1a-ii) Since $U_{n+1} = \frac{(n+1)^2}{(3n+4)!}$, hence $\frac{U_{n+1}}{U_n} = \frac{(n+1)^2}{(3n+4)!}\frac{(3n+1)!}{n^2} = \frac{(n+1)^2}{(3n+4)(3n+3)(3n+2)n^2}$, so $\lim_{n\to\infty}\frac{U_{n+1}}{U_n} = 0 < 1$, therefore $\sum_{n=1}^{\infty}\frac{n^2}{(3n+1)!}$ is convergent,

1a-iii) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n^{8/5}$ is divergent, therefore $\sum_{n=1}^{\infty}\frac{n^2}{\sqrt[5]{n^2+1}}$ is convergent as $n^{8/5} > \frac{n^2}{\sqrt[5]{n^2+1}}$.

1b-i) By separation method, we get $xe^x\,dx = \cos^2 y\,dy$, therefore $xe^x - e^x = [y + (\sin 2y)/2]/2$

1b-ii) Since $\frac{dy}{dx} = \frac{x+y+3}{x-y+1}$ is non homogeneous equation. To solve this differential equation, we have to follow these steps

(1) We have to get the point of intersection between x+y+3=0, x-y+1=0 which is (-2,-1),

(2) Put x=X-2, y=Y-1, dx=dX,dy=dY in the above differential equation, then $\frac{dY}{dX} = \frac{X+Y}{X-Y}$, so it is a homogeneous equation,

(3) Put Y=vX, and dY=vdX+Xdv, therefore $\frac{vdX+Xdv}{dX} = \frac{X+vX}{X-vX} = \frac{1+v}{1-v}$

(4) Integrate $\frac{dX}{X} = \frac{(1-v)dv}{1+v^2}$, then put X=x+2, $v = \frac{Y}{X} = \frac{y+1}{x+2}$ so that the solution of the differential equation is $\mathrm{Ln}(x+2) = \tan^{-1}(\frac{y+1}{x+2}) - \frac{1}{2}\ln(\frac{(y+1)^2+(x+2)^2}{(x+2)^2})$ +C

1b-iii) put z = 1/y $\Rightarrow$ z` + tanx z = - $\cos^3 x \Rightarrow$ z/cosx = -[x + (sin2x)/2]/2$\Rightarrow$ 1/ycosx = -[x + (sin2x)/2]/2

1c) $\frac{\partial}{\partial\alpha}(x\cos\alpha + y\sin\alpha = P)$, therefore $-x\sin\alpha + y\cos\alpha = 0$, thus $\tan\alpha = y/x$, so $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$, hence envelope is $x^2 + y^2 = P^2$.

2a-i) Since $U_n = \frac{3^n}{(n^2+1)(x-2)^n}$, and $U_{n+1} = \frac{3^{n+1}}{((n+1)^2+1)(x-2)^{n+1}}$, hence the ratio $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{3^{n+1}(n^2+1)(x-2)^n}{3^n((n+1)^2+1)(x-2)^{n+1}}\right| = \left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right|$,therefore

$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right| = \lim_{n\to\infty}\left|\frac{3}{(x-2)}\right| < 1$ to be convergent, hence $|x-2| > 3$, thus x > 5 or x < -1 is the interval of convergence.

2a-ii) Since $U_n = \frac{(-1)^n (x-1)^{2n}}{2n!}$, and $U_{n+1} = \frac{(-1)^{n+1} (x-1)^{2n+2}}{(2n+2)!}$, hence the ratio $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(-1)^{n+1} (x-1)^{2n+2}\ 2n!}{(-1)^n (x-1)^{2n} (2n+2)!}\right| = \left|\frac{(x-1)^2}{(2n+2)(2n+1)}\right|$, thus $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^2}{(2n+2)(2n+1)}\right| = 0$ hence series is convergent for all x.

2b) f(x,y,z) = xyz, $\phi(x,y,z) = 16x^2 + 4y^2 + 9z^2 = 144$ and $f_x = \lambda\phi_x$, $f_y = \lambda\phi_y$ and $f_z = \lambda\phi_z$, therefore $yz = \lambda(32x)$, $xz = \lambda(8y)$ and $xy = \lambda(18z)$, thus y = 2x , z = 4/3 x, thus $x = \sqrt{3}$, $y = 2\sqrt{3}$, $z = 4/3\sqrt{3}$, so the largest volume = $8\sqrt{3}$.

2c-i) $(y + \ln(x))dx + (x+y^2)\,dy = 0$ is exact D.E. since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$, thus $\frac{\partial f}{\partial x} = M(x,y) = (y + \ln(x)) \Rightarrow f(x,y) = xy + x\mathrm{Ln}\,x - x + \phi(y)$, thus $\frac{\partial f}{\partial y} = x + \phi'(y) = x+y^2$, hence $\phi(y) = y'' + 2y' +$

2c-ii) $y'' + y = 1 + \tan x$ has homogeneous and particular solution so that the characteristic equation is $m^2 + 1 = 0 \Rightarrow m = -i, i$, thus $y_H = (c_1 \cos x + c_2 \sin x)$ and so the particular solution is: $y_P = u_1(x)\cos x + u_2(x)\sin x$, and $y_1(x) = \cos x$, $y_2(x) = \sin x$ where $u_1(x) = -\int \frac{y_2 g(x)}{W(y_1, y_2)}dx$, $u_2(x) = \int \frac{y_1 g(x)}{W(y_1, y_2)}dx$, where $W(y_1, y_2) = y_2' y_1 - y_2 y_1' = 1$, $g(x) = 1 + \tan x$, therefore:

$$u_1(x) = -\int \frac{\sin x\,(1+\tan x)}{1}dx = -\cos x + \ln(\sec x + \tan x) - \sin x,$$

$$u_2(x) = \int \frac{\cos x\,(1+\tan x)}{1}dx = \sin x - \cos x$$

2c-iii) $y'' + 2y' + 2y = e^x \sin^2(2x)$ has homogeneous and particular solution so that the characteristic equation is $m^2 + 2m + 2 = 0 \Rightarrow m = -1 \pm i$, thus $y_H = e^{-x}(c_1\cos x + c_2 \sin x)$and so the particular solution is

$$y_P = \frac{1}{D^2+2D+2} e^x \sin^2 x = \frac{1}{D^2+2D+2}\left(\frac{e^x}{2}\right)(1-\cos 2x) = \frac{e^x}{2}\left[\frac{1}{5} - \frac{(8\sin 2x + \cos 2x)}{65}\right]$$

1b-vi) Since $\frac{dy}{dx} = \frac{x+y+3}{x-y+1}$ is non homogeneous equation. To solve this differential equation, we have to follow these steps

(1) We have to get the point of intersection between x+y+3=0, x-y+1=0 which is (-2,-1),

(2) Put x=X-2, y=Y-1, dx=dX,dy=dY in the above differential equation, then $\frac{dY}{dX} = \frac{X+Y}{X-Y}$, so it is a homogeneous equation,

(3) Put Y=vX, and dY=vdX+Xdv, therefore $\frac{vdX+Xdv}{dX} = \frac{X+vX}{X-vX} = \frac{1+v}{1-v}$

(4) Integrate $\frac{dX}{X} = \frac{(1-v)dv}{1+v^2}$, then put X=x+2, $v = \frac{Y}{X} = \frac{y+1}{x+2}$ so that the solution of the differential equation is $\mathrm{Ln}(x+2) = \tan^{-1}\left(\frac{y+1}{x+2}\right) - \frac{1}{2}\ln\left(\frac{(y+1)^2+(x+2)^2}{(x+2)^2}\right) + C$

2a) Since $f(x, y) = \tan^{-1}\left(\frac{x+y}{x-y}\right)$, therefore $f_x = \frac{-y}{x^2+y^2}$, $f_y = \frac{x}{x^2+y^2}$, $f_{xx} = \frac{2xy}{(x^2+y^2)^2}$, $f_{xx} = \frac{2xy}{(x^2+y^2)^2}$, $f_{yy} = \frac{-2xy}{(x^2+y^2)^2}$, and $f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}$. At (0,1) , therefore $f(0,1) = -\frac{\pi}{4}$, $f_x = -1$, $f_y = 0$, $f_{xx} = f_{yy} = \frac{2xy}{(x^2+y^2)^2} = 0$, $f_{xy} = 1$, then by substituting in Taylor formula, we get: $f(x, y) = -\frac{\pi}{4} - x + x(y-1)$

2b-i) Since $U_n = \dfrac{3^n}{(n^2+1)(x-2)^n}$, and $U_{n+1} = \dfrac{3^{n+1}}{((n+1)^2+1)(x-2)^{n+1}}$, hence the

ratio $\left|\dfrac{U_{n+1}}{U_n}\right| = \left|\dfrac{3^{n+1}(n^2+1)(x-2)^n}{3^n((n+1)^2+1)(x-2)^{n+1}}\right| = \left|\dfrac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right|$,therefore

$\lim_{n\to\infty}\left|\dfrac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\dfrac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right| = \lim_{n\to\infty}\left|\dfrac{3}{(x-2)}\right| < 1$ to be convergent, hence $|x-2| > 3$, thus $x > 5$ or $x < -1$ is the interval of convergence.

2b-ii) Since $U_n = \dfrac{(-1)^n}{(2n+1)!\,x^{2n+1}}$, and $U_{n+1} = \dfrac{(-1)^{n+1}}{(2n+3)!\,x^{2n+3}}$, hence the

ratio $\left|\dfrac{U_{n+1}}{U_n}\right| = \left|\dfrac{(2n+1)!\,x^{2n+1}}{(2n+3)!\,x^{2n+3}}\right| = \left|\dfrac{1}{(2n+3)(2n+2)x^2}\right|$,therefore $\lim_{n\to\infty}\left|\dfrac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\dfrac{1}{(2n+3)(2n+2)x^2}\right| = 0$ is convergent for all x.

2b-iii) Since $U_n = \dfrac{(x-2)^n}{n^3+1}$, and $U_{n+1} = \dfrac{(x-2)^{n+1}}{(n+1)^3+1}$, hence the

ratio $\left|\dfrac{U_{n+1}}{U_n}\right| = \left|\dfrac{[(n)^3+1](x-2)^{n+1}}{(x-2)^n[(n+1)^3+1]}\right| = \left|\dfrac{[(n)^3+1]}{[(n+1)^3+1](x-2)}\right|$,and $\lim_{n\to\infty}\left|\dfrac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\dfrac{[(n)^3+1]}{[(n+1)^3+1](x-2)}\right| < 1$,

Thus $|x-2| > 1$ so that $x > 3$, $x < 1$

3a) We have to get f_x, f_y, f_{xx}, f_{xy}, f_{yy} such that $f_x = \dfrac{1}{x+y} - \dfrac{1}{x-y}$, $f_y = \dfrac{1}{x+y} + \dfrac{1}{x-y}$,

$f_{xx} = \dfrac{-1}{(x+y)^2} + \dfrac{1}{(x-y)^2}$ $f_{yy} = \dfrac{-1}{(x+y)^2} + \dfrac{1}{(x-y)^2}$, $f_{xy} = \dfrac{-1}{(x+y)^2} - \dfrac{1}{(x-y)^2}$

Therefore: at (0, 1), $f(0, 1) = 0$, $f_x = 2$, $f_y = 0$, $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = -2$, therefore $f(x,y) = f(0, 0) + \dfrac{1}{1!}\big(f_x(0,0)(x-0) + f_y(0,0)(y-0)\big) + \dfrac{1}{2!}\big(f_{xx}(0,1)(x-0)^2 + 2(x-0)(y-1)f_{xy}(0,1) + f_{yy}(0,1)(y-1)^2\big)$, therefore $f(x,y) = 2x + x(y-1)$

3b-i) Put $x = r\cos\theta$, $y = r\sin\theta$, therefore $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^{3} r^2\,dr\,d\theta = 9\pi$

3b-ii) Since the contour integration is the line joining (0,2) and (1,4) such that $y = 2x + 2$, therefore

$dy = 2dx$, hence $\int_c \sin(\pi y)\,dx + yx^2 dy = \int_0^1 \sin(2\pi x)\,dx + (2x+2)x^2(2)dx$

$= \int_0^1 \sin(2\pi x)\,dx + 4(x^3+x^2)dx = \dfrac{-\cos(2\pi x)}{2\pi} + \left(x^4 + \dfrac{4}{3}x^3\right)\Big|_0^1 = \dfrac{7}{3}$

3c) Volume $= u.(v\times w) = \begin{vmatrix}1 & 0 & 2\\ 0 & 2 & 3\\ 0 & 1 & 3\end{vmatrix} = 3.$