Benha University Faculty of Engineering (Shoubra) Engineering Mathematics and Physics Department		Functions of advanced complex variable Final exam 7th of June 2014 Qualified year EMP 408 Time allowed: 3 hours Scores : 200

Answer the following Questions

Question 1 [60 Marks]

- **Derive** Cauchy Riemann equations in Polar form.
- **State** and **prove** Cauchy's Theorem
- **Discuss** briefly the following terms:

Laurent Series –Analytic function –Removable singularities - Essential singularities- Residues – Pole- – Calculation of Residues – Linear and Bilinear transformations.

- **Expand** the function $f(z)=\dfrac{z}{(z-1)(2-z)}$, a) $|z|<1$, b) $1<|z|<2$, c) $0<|z-2|<1$,

then deduce the residues.

Question 2 [80 Marks]

Choose the correct answer

- If $f(z) = e^x\cos(ay) + i\ e^x\sin(y-b)$ is differentiable at every point, then a and b values equal to (0 and i, 1 and 0, 1 and 2, i and -i)
- The harmonic conjugate of $x^2 - y^2 + y$ is ($2xy$, $2xy+x$, $2xy-x$, $x^2 - 2xy$)
- $f(z) = \dfrac{a}{2} r\cos\theta + i(r\sin\theta + 2)$ is harmonic if (a=0, a = 1, a = 2, a = 3)
- The residue of $\dfrac{e^z}{z^2(z^2+9)}$ at z = 0 is (0,1/3 , 1/9, 1/27)
- $\displaystyle\int_{|z|=2} \tan z\, dz$ = ($2\pi i$, $-2\pi i$, πi, $4\pi i$, $-4\pi i$)
- If $\sin^{-1}(\cos\theta + i\sin\theta) = x+iy$, then $\cos\theta$ is (sinx sinhy, cosx sinhy, cosx coshy, sinx coshy)
- If sinz = 2, then e^{iz} = ($i\sqrt{3}$, $i(1+\sqrt{3})$, $i(1-\sqrt{3})$, $i(2+\sqrt{3})$)

Question 3 [60 Marks]

Evaluate the following integrals

$$\oint_{|z|=} \frac{e^{2z}}{(z^2+9)^2(z^2+2z+2)}\,dz\ , \qquad \int_{1+i}^{2i} e^z \cos 2z \sin 5z\, dz \qquad \oint_{|z|=3} \frac{[\sin \pi z^2 + \cos \pi z^2]}{(z-1)(z-2)}\,dz$$

$$\oint_{|z-i|=2} \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz \qquad \oint_{|z-2|=\frac{1}{2}} \frac{z}{(z-1)(z-2)}\,dz \qquad \oint_{|z|=2} [\frac{e^z}{z^3} + \frac{z^4}{(z+i)^2}]dz$$

Dr. eng. Khaled El Naggar

Model answer

Answer of Question 1

- **In Polar form**

Cauchy Riemann equations are $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$, where x=r cos$\theta$, y = r sin$\theta$.

Proof

Since $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}(\cos\theta) + \frac{\partial v}{\partial y}(\sin\theta)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

But $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, therefore $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$.

- **Cauchy theorem**

If a function f(z) is analytic inside and on a simple closed curve C and f`(z) is continuous inside and on the curve C, then $\oint_c f(z)\,dz = 0$.

Proof

$\oint_c f(z)\,dz = \oint_c u\,dx - v\,dy + i\oint_c v\,dx + u\,dy$

By Green `s theorem, we can get

$\oint_c f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$, but f(z) is analytic inside and on a simple closed curve C, therefore f(z) satisfy Cauchy Riemann, i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, hence $\oint_c f(z)\,dz = 0$

- **Laurent Series:** If f(z) has a pole of order n at z = a , but is analytic at every other point inside and on a circle c with center at a , then $(z-a)^n f(z)$ is analytic at all points inside and on c and has Taylor series about z= a so that :

$$f(z)=\frac{a_{-n}}{(z-a)^n}+\frac{a_{-n+1}}{(z-a)^{n-1}}+....+\frac{a_{-1}}{(z-a)}+a_0+a_1(z-a)+a_2(z-a)^2+....$$ called Laurent Series for f(z) and

$a_0+a_1(z-a)+a_2(z-a)^2+.....$ is called analytic part, the remaining is called principle part, z = a is called pole of order n.

Analytic function: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R.

Removable singularities: If a single–valued function f(z) is not defined at z = a , but $\lim_{z\to a}\frac{f(z)}{z-a}$ exist, z = a called removable singularity

Essential singularities: If f(z) is single–valued, then any singularity which is not a pole or removable singularity is called essential singularity.

Residues: Let f(z) be single-valued and analytic inside and on a circle c except at the point z = a chosen as center of c , then the Laurent series about z = a is given by $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-a)^n$, where a_{-1} is the residue of f(z)

Calculation of Residues:

1st case: (Simple pole)

The residue of the function f(z) at z =a which is simple pole (pole of order one) is given by

$$\operatorname*{Res}_{z=a}=a_{-1}=\lim_{z\to a}(z-a)f(z)$$

2nd case: (pole of order n)

The residue of the function f(z) at z =a which is pole of order n is given by

$$\operatorname*{Res}_{z=a}=a_{-1}=\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z)$$

$$f(z)=\frac{z}{(z-1)(2-z)}=\frac{1}{(z-1)}+\frac{2}{(2-z)}=-\frac{1}{(1-z)}+\frac{1}{(1-\frac{z}{2})}=2+\frac{3z}{2}+\frac{5z^2}{4}+\ldots \qquad |z|<1$$

$$f(z)=\frac{z}{(z-1)(2-z)}=\frac{1}{(z-1)}+\frac{2}{(2-z)}=\frac{1}{z(1-\frac{1}{z})}+\frac{1}{(1-\frac{z}{2})}=\frac{1}{z}[1+\frac{1}{z}+(\frac{1}{z})^2+\ldots]+1+\frac{z}{2}+(\frac{z}{2})^2+\ldots$$

$1<|z|<2$. Therefore the residue at z = 1 equal 1

$$f(z)=\frac{z}{(z-1)(2-z)}=\frac{1}{(z-1)}+\frac{2}{(2-z)}=\frac{1}{z(1-\frac{1}{z})}-\frac{2}{z(1-\frac{2}{z})}=\frac{1}{z}[1+\frac{1}{z}+(\frac{1}{z})^2+\ldots]+\frac{2}{z}[1+\frac{2}{z}+(\frac{2}{z})^2+\ldots] \; .$$

$|z|>2$ Therefore the residue at z = 2 equal 2.

Answer of Question 2

- If $f(z) = e^x\cos(ay) + i\,e^x\sin(y-b)$ is differentiable at every point, then a and b values equal to (0 and i, <u>1 and 0</u>, 1 and 2, i and -i)
- The harmonic conjugate of $x^2 - y^2 + y$ is (2xy, 2xy+x, <u>2xy-x</u>, $x^2 - 2xy$)
- f(z) = $\frac{a}{2} r\cos\theta + i(r\sin\theta + 2)$ is harmonic if (a=0, a = 1, <u>a = 2</u>, a = 3)
- The residue of $\frac{e^z}{z^2(z^2+9)}$ at z = 0 is (0,1/3 , <u>1/9</u>, 1/27)
- $\int_{|z|=2} \tan z\,dz$ = ($2\pi i$, $-2\pi i$, πi, $4\pi i$, <u>$-4\pi i$</u>)
- If $\sin^{-1}(\cos\theta + i\sin\theta)$ x+iy, then $\cos\theta$ is (sinx sinhy, cosx sinhy, cosx coshy, <u>sinx coshy</u>)
- If sinz = 2, then e^{iz} = ($i\sqrt{3}$, <u>$i(1+\sqrt{3})$</u>, $i(1-\sqrt{3})$, $i(2+\sqrt{3})$)

Answer of Question 3

$$\oint_{|z|=} \frac{e^{2z}}{(z^2+9)^2(z^2+2z+2)}dz = 2\pi i\,[\mathop{Res}_{z=3i}+\mathop{Res}_{z=-3i}+\mathop{Res}_{z=-1+i}+\mathop{Res}_{z=-1-i}]$$, **where**

$$\mathop{Res}_{z=3i} = \lim_{z\to 3i}\frac{d}{dz}[z-3i]^2\frac{e^{2z}}{(z^2+9)^2(z^2+2z+2)} = \lim_{z\to 3i}\frac{d}{dz}\frac{e^{2z}}{(z+3i)^2(z^2+2z+2)}$$

$$\underset{z=-3i}{\text{Res}} = \lim_{z\to -3i} \frac{d}{dz}[z+3i]^2 \frac{e^{2z}}{(z^2+9)^2(z^2+2z+2)} = \lim_{z\to -3i} \frac{d}{dz} \frac{e^{2z}}{(z-3i)^2(z^2+2z+2)}$$

$$\underset{z=-1+i}{\text{Res}} = \lim_{z\to -1+i} [z-(-1+i)] \frac{e^{2z}}{(z^2+9)^2(z^2+2z+2)} = \lim_{z\to -1+i} \frac{e^{2z}}{(z^2+9)^2[z-(-1-i)]}$$

$$\underset{z=-1-i}{\text{Res}} = \lim_{z\to -1-i} [z-(-1-i)] \frac{e^{2z}}{(z^2+9)^2(z^2+2z+2)} = \lim_{z\to -1-i} \frac{e^{2z}}{(z^2+9)^2[z-(-1+i)]}$$

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$$\oint_{|z|=3} \frac{[\sin \pi z^2 + \cos \pi z^2]}{(z-1)(z-2)}\, dz = 2\pi i\ [\underset{z=1}{\text{Res}} + \underset{z=2}{\text{Res}}]$$**, where**

$$\underset{z=1}{\text{Res}} = \lim_{z\to 1} [z-1] \frac{[\sin \pi z^2 + \cos \pi z^2]}{(z-1)(z-2)} = 1, \quad \underset{z=2}{\text{Res}} = \lim_{z\to 2} [z-2] \frac{[\sin \pi z^2 + \cos \pi z^2]}{(z-1)(z-2)} = 1$$

Therefore $\oint_{|z|=3} \frac{[\sin \pi z^2 + \cos \pi z^2]}{(z-1)(z-2)}\, dz = 4\pi i$

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$$\oint_{|z-i|=2} \frac{z^2-2z}{(z+1)^2(z^2+4)}\, dz = 2\pi i\ [\underset{z=-1}{\text{Res}} + \underset{z=2i}{\text{Res}}]$$

$$\underset{z=-1}{\text{Res}} = \lim_{z\to 1} \frac{d}{dz}[z+1]^2 \frac{z^2-2z}{(z+1)^2(z^2+4)} = \lim_{z\to 1} \frac{d}{dz} \frac{z^2-2z}{(z^2+4)},$$

$$\underset{z=2i}{\text{Res}} = \lim_{z\to 2i} [z-2i] \frac{z^2-2z}{(z+1)^2(z^2+4)} = \lim_{z\to 2i} \frac{z^2-2z}{(z+1)^2(z+2i)}$$

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$$\oint_{|z-2|=\frac{1}{2}} \frac{z}{(z-1)(z-2)}\, dz = \oint_{|z-2|=\frac{1}{2}} \frac{z/z-1}{(z-2)}\, dz = 4\pi i$$

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$$\oint_{|z|=2} [\frac{e^z}{z^3} + \frac{z^4}{(z+i)^2}] dz = \pi i + 2\pi i(-4i) = [8+i]\pi$$