Benha University
Faculty of Engineering- Shoubra
Energy and Sustainable Energy program

Final Exam
Mathematics 2
Date: 03-6-2014
Code: EMP102
Duration : **3 hours**

Answer three questions from the following: **[20 Marks]**

1) a) Find the perpendicular distance between the following straight lines:

$3x + 4y - 8 = 0$ and $3x + 4y + 10 = 0$.

b) Show that (4 , 2), (13 , 14), (–4 , 8) are the vertices of a right triangle and find the area of the triangle by two methods.

c) Find the equation of a straight line, which passes through the point (4, 3), and forms with the co-ordinate axes a triangle of area 24 sq. units.

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2)a) Show that the lines bisecting the angles between the bisectors of the angle made by $a x^2 + 2 h x y + b y^2 = 0$, are given by $(a - b)(x^2 - y^2) + 4 h x y = 0$ and prove that the pairs of lines $(x^2 - y^2)/(a - b) = x y / h$ and $(a - b)(x^2 - y^2) + 4 h x y = 0$, are such that each represents the bisectors of the other.

b) Write down the equation to the bisectors of the angles between the line-pair $3x^2 + 2xy + 3y^2 = 0$. Can you draw the straight lines? or the bisectors ? If not, Why not ?

c) For what values of k does $3 x^2 + 7x y + 2 y^2 + x - 3 y - k = 0$, represent pair of straight lines? Find these equations, the point of intersection and the angle between them.

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3)a) Find the equation of the circle, which touches the x- axis at the point (3,0) and passes through the point (1,2).

b) Find the equation of the circle which passes through the points (1, 6) , (5,0) and whose centre lies on the line $5x + 12 y = 25$.

c) Find the equation of tangent and normal to the circle $x^2 + y^2 - 2 x - 4 y + 3 = 0$, at the point (2, 3).

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4)a) Find the pole of the straight line $x + 9 y - 28 = 0$, with respect to the circle $2x^2 + 2y^2 + 5x - 3 y - 7=0$.

b) Find the equation to the parabola, whose focus is the point (2 , 3) and whose directrix is the straight line $x - 4 y + 3 = 0$. Find also the length of its latus rectum.

c) Prove that the locus of the mid-point of the chords of the parabola $y^2 = 4 a x$ which pass through the vertex is the parabola $y^2 = 2ax$.

Answer two of the following questions **[20 Marks]**

1- Evaluate the following integrals

a) $\int \frac{\sin 2x}{1+\sin^2 x} dx$ b) $\int \frac{dx}{(x^2+4x+5)^2}$ c) $\int \frac{x^2+5}{x^2+3x+2} dx$

2-a) Solve the differential equation $y`` = x + \cos x$, $y(0)=2$, $y`(0)=4$

b) Find area bounded by two curves $f(x) = x$ and $g(x) = x^2$

3- Evaluate the following integrals

a) $\int \frac{(x+4)dx}{\sqrt{x^2+6x+10}}$ b) $\int_{-\pi/4}^{\pi/4} x^2[\cos x + \sin x]\, dx$ c) $\int [1+\frac{1}{x}] \cos(x + \ln x)\, dx$

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Model answer

Answer only three questions from the following:

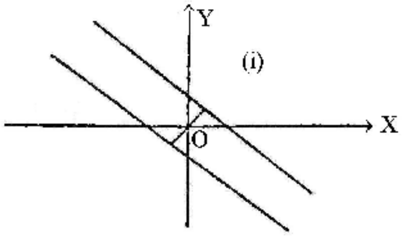
1) a) Find the perpendicular distance between the following straight lines:

$3x + 4y – 8 = 0$ and $3x + 4y + 10 = 0$.

a) In the first equation $3x + 4y – 8 = 0$, let $x = 0$ then $y = 2$,

then the point (0 , 2) will lies on $3x + 4y – 8 = 0$ (cf. Figure 1.21).

Hence the required distance between them is $p = (8 +10)/5=18/5$.



b) Show that (4 , 2), (13 , 14), (–4 , 8) are the vertices of a right triangle and find the area of the triangle by two methods.

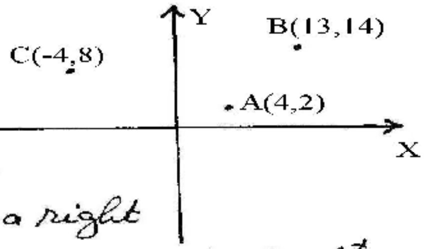
Let $A(4,2)$, $B(13,14)$, $C(-4,8)$.

$AB^2 = (13-4)^2 + (14-2)^2 = 81 + 144 = 225$,

$Bc^2 = (-4-13)^2 + (8-14)^2 = (17)^2 + 6^2 = 289+36=325$

$CA^2 = (-4-4)^2 + (8-2)^2 = 8^2 + 6^2 = 64 + 36 = 100$

It is clear that $AB^2 + CA^2 = BC^2$ $\therefore \Delta ABC$ is a right triangle at A. The area of $\Delta ABC = \frac{1}{2} AB \cdot CA = \frac{1}{2}(15)(10) = 75$ sq. units

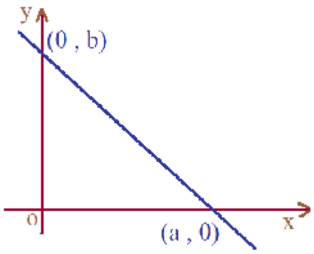


$$\frac{1}{2}\begin{Vmatrix} 1 & 4 & 2 \\ 1 & 13 & 14 \\ 1 & -4 & 8 \end{Vmatrix} = 75 \text{ square units.}$$

c) **Find** the equation of a straight line, which passes through the point (4, 3), and forms with the co-ordinate axes a triangle of area 24 sq. units.

c) Let the equation of the straight line be $x/a + y/b = 1$.
It passes through (4 , 3) , then $4/a + 3/b =1$,
i.e. $4b + 3a = a b$, (1)
The area of the triangle $= 24 = ab/2$, i.e. $a b = 48$ (2)
From (1) and (2) , we get $4b + 3a = 48$. (3)

Multiplying (3) by a, we get $4ab+3a^2= 48 a$



i.e. $3a^2 – 48 a + 192 = 0$, hence $a^2 – 16 a + 64 = 0$, i.e. $(a – 8)^2 = 0$.
$\therefore$ $a = 8$ and therefore $b = 6$. Equation of the line $x/8 + y/6 = 1$.

2)a) Show that the lines bisecting the angles between the bisectors of the angle made by $a\,x^2 + 2\,h\,x\,y + b\,y^2 = 0$, are given by $(a - b)(x^2 - y^2) + 4\,h\,x\,y = 0$ and prove that the pairs of lines $(x^2 - y^2)/(a - b) = x\,y\,/\,h$ and $(a - b)(x^2 - y^2) + 4\,h\,x\,y = 0$, are such that each represents the bisectors of the other.

Bisectors of $a\,x^2 + 2\,h\,x\,y + b\,y^2 = 0$ are

$$\frac{x^2 - y^2}{a - b} = \frac{x\,y}{h}, \text{ i.e. } h\,x^2 - (a - b)\,x\,y - h\,y^2 = 0. \quad (1)$$

Bisectors of the pairs of lines (1) are

$$\frac{x^2 - y^2}{h - (-h)} = \frac{x\,y}{-(a - b)/2} \text{ i.e. } \frac{x^2 - y^2}{2h} = \frac{2x\,y}{-(a - b)}$$

i.e. $(a - b)(x^2 - y^2) + 4\,h\,x\,y = 0$. (2)

Now the bisectors of (1) are given by (2) and the bisectors of (2) are given by

$$\frac{x^2 - y^2}{2(a - b)} = \frac{x\,y}{2h} \text{ or } \frac{x^2 - y^2}{a - b} = \frac{x\,y}{h}, \text{which is (1).}$$

2)b) Write down the equation to the bisectors of the angles between the line-pair $3x^2 + 2xy + 3y^2 = 0$. Can you draw the straight lines? or the bisectors ? If not, Why not ?

Equation of the bisectors is $\frac{x^2 - y^2}{3-3} = \frac{xy}{2}$, which means that $x^2 - y^2 = 0$. Since $3x^2 + 2xy + 3y^2 = 0 \Rightarrow 3\frac{y^2}{x^2} + 2\frac{y}{x} + 3 = 0$

i.e. $\frac{y}{x} = \frac{-2 \pm \sqrt{4-36}}{(2)(3)} = \frac{-2 \pm \sqrt{-32}}{6} = -\frac{1}{3} \pm \frac{2}{3}\sqrt{2}\,i$

i.e. two imaginary lines, we can not draw these lines, but the bisectors is $(x-y)(x+y) = 0$ and we can draw it.

2) c) **For what** values of k does $3\,x^2 + 7x\,y + 2\,y^2 + x - 3\,y - k = 0$, represent pair of straight lines? **Find** these equations, the **point of intersection** and the **angle** between them.

The condition that the equation shall represent two lines is

$$\begin{vmatrix} 2 & -7/2 & 11/2 \\ -7/2 & 3 & -13/2 \\ 11/2 & -13/2 & c \end{vmatrix} = 0 \quad \Rightarrow c - 12 = 0, \text{ i.e. } c = 12.$$

i.e. It must be added +2.Then

$2\,x^2 - 7x\,y + 3\,y^2 + 11x - 13\,y + 12 = 0.$

Factorising, we get $(2x - y + p)(x - 3y + q) = 0$,

$\Rightarrow p + 2q = 11,\ 3p + q = 13,\ p\,q = 12.$

Solving, we get $p = 3$, $q = 4$; i.e. $(2x - y + 3)(x - 3y + 4) = 0$.

It is easy to see that the point of intersection is $(-1, 1)$.

$$\tan\theta = \pm 2\frac{\sqrt{49/4 - 6}}{2 + 3} = \pm 1, \text{ i.e. } \theta = \pi/4 \text{ or } 3\pi/4.$$

3)a) Find the equation of the circle, which touches the x- axis at the point (3,0) and passes through the point (1,2).

In case the circle touches the axis of x, $g^2 = c$ and $-g = 3$, i.e. $g = -3$ and $c = 9$.
Since the circle passes through (1,2), then $1 + 4 + 2g + 4f + c = 0$, or $4f = -8$,
i.e. $f = -2$. Thus $x^2 + y^2 - 6x - 4y + 9 = 0$, is the required equation.

b) Find the equation of the circle which passes through the points (1, 6), (5,0) and whose centre lies on the line $5x + 12y = 25$.

Suppose that the equation to the circle is
$x^2 + y^2 + 2gx + 2fy + c = 0$, its centre $(-g, -f)$ lies on the given line
i.e. $-5g - 12f = 25$. (1)
Since it passes through the given points, then
$25 + 10g + c = 0$, (2) and $37 + 2g + 12f + c = 0$. (3)
Solving equations (l), (2) and (3), we get
$g = -1$, $f = -5/3$, $c = -15$.
Therefore $x^2 + y^2 - 6x - 10y - 45 = 0$, is the required equation.

c) Find the equation of tangent and normal to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$, at the point (2, 3).

The equation of the tangent is
$2x + 3y - (x + 2) - 2(y + 3) + 3 = 0$, or $x + y = 5$, its slope is -1.
Slope of the normal is l, and then its equation is
$(y - 3)/(x - 2) = 1$, or $y = x + 1$.

4)a) Find the pole of the straight line $x + 9y - 28 = 0$, with respect to the circle
$2x^2 + 2y^2 + 5x - 3y - 7 = 0$.

If (a, b) be the required point, the line $x + 9y - 28 = 0$, (i)
must coincide with the polar of (a, b), whose equation is
$2ax + 2by + (5/2)(x + a) - (3/2)(y + b) - 7 = 0$.
i.e. $(4a + 5)x + (4b - 3)y + 5a - 3b - 14 = 0$. (ii)
Since (i) and (ii) are the same, we have

$$\frac{4a+5}{1} = \frac{4b-3}{9} = \frac{5a-3b-14}{-28}$$

Hence $9a - b = -12$, $117a - 3b = -126$.
Solving these equations we have $a = -1$ and $b = 3$, so that the required point is
(–1 ,3).

b) Find the equation to the parabola, whose focus is the point (2 , 3) and whose directrix is the straight line $x - 4y + 3 = 0$. Find also the length of its latus rectum.

The equation of the parabola is
$(x - 2)^2 + (y - 3)^2 = [x - 4y + 3]^2/(1^2+4^2)$, squaring
i.e. $17[x^2 + y^2 - 4x - 6y + 13] = [x^2+16y^2+9-8xy+6x-24y]$
i.e. $16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$,
or $(4x + y)^2 = 74x + 78y - 212$.
Length of the latus rectum =
=2(the distance of the focus from the directrix)

$$= 2 \,.\, \frac{|2-12+3|}{\sqrt{17}} = \frac{14}{\sqrt{17}} .$$

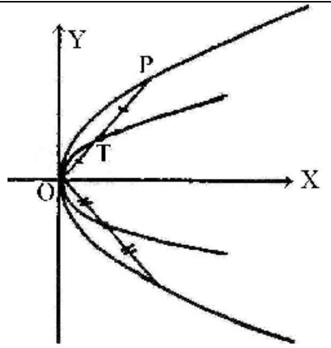
c) Prove that the locus of the mid-point of the chords of the parabola $y^2 = 4ax$ which pass through the vertex is the parabola $y^2 = 2ax$.

Let T(h , k) be the mid-point of the chord OP through the origin O(0,0) and

P(at^2, 2at), then h = ½ at^2, k = a t . Eliminating t, we get

h = ½ a k^2 / a^2 or k^2 = 2ah.

Then the locus of (h , k) is y^2 = 2 a x, which is a parabola one half the size of the original, and with the same vertex, axis and tangent at the vertex.



1-a) $\int \frac{\sin 2x}{1+\sin^2 x} dx = \ln(1+\sin^2 x) + c$

1-b) put x+2 = $\tan\theta \Rightarrow dx = \sec^2\theta \; d\theta$, therefore $\int \frac{dx}{(x^2+4x+5)^2} = \int \frac{dx}{[(x+2)^2+1]^2} =$

$$\int \frac{\sec^2\theta d\theta}{[(\tan\theta)^2+1]^2} = \int \cos^2\theta d\theta = \frac{1}{2}\int[1+\cos 2\theta]d\theta = \frac{\theta+\sin\theta\cos\theta}{2}$$

$$= \frac{\tan^{-1}(x+2)}{2} + \frac{(x+2)}{2\sqrt{x^2+4x+5}}$$

1-c) The degree of numerator must be less than that of denominator, so we have to make long division

$$\begin{array}{r|l} & 1 \\ \hline x^2 + 3x + 2 & x^2+5 \\ & \underline{x^2+3x+2} \\ & -3x + 3 \end{array}$$

$$\frac{x^2+5}{x^2+3x+2} = 1 + \frac{-3x+3}{x^2+3x+2}$$

$$\int \frac{x^2+5}{x^2+3x+2} dx = \int (1+\frac{-3x+3}{x^2+3x+2})\, dx = \int (1+\frac{-3x+3}{(x+2)(x+1)})\, dx = x + \int(\frac{A}{(x+1)} + \frac{B}{(x+2)})dx$$

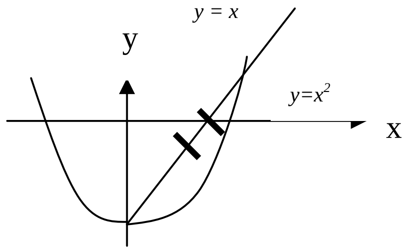
By using partial fraction, we will get A= 6, B = -9

$$\int \frac{x^2+5}{x^2+3x+2} dx = x + \int(\frac{6}{(x+1)} - \frac{9}{(x+2)})dx = x + 6Ln(x+1) - 9Ln(x+2) + c$$

2- y\`(x)= $x^2/2$ + sinx +c , but at y\`=4, x = 0, therefore c = 4, therefore y\`(x)= $x^2/2$ + sinx +4, integrate, we will get y(x)= $x^3/6$ – cosx + 4x+d , but at y=2 , x=0, therefore d= 3, thus the solution is y(x)= $x^3/6$ – cosx + 4x+3

b)

$$A = \int_0^1 [x - x^2]dx = \frac{x^2}{2} - \frac{x^3}{6}\Big|_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$



3-a) By completing square, we get $\int \frac{(x+4)dx}{\sqrt{x^2+6x+10}} = \int \frac{(x+3+1)dx}{\sqrt{(x+3)^2+1}}$

$$= \frac{1}{2}\int \frac{2(x+3)dx}{\sqrt{(x+3)^2+1}} + \int \frac{dx}{\sqrt{(x+3)^2+1}} = \sqrt{(x+3)^2+1} + \sinh^{-1}(x+3)$$

3-c) $\int [1+\frac{1}{x}]\cos(x+\ln x)\,dx = \sin(x+\ln x)$