Benha University Faculty of Engineering (Shoubra) Engineering Mathematics and Physics Department		Functions of advanced complex variable Final exam 6-6-2015 Qualified year EMM 408 Time allowed: 3 hours Scores : 200

Question 1 **[100]**

i- Evaluate the following integrals: $\int_{1+i}^{2+8i} [x^2 + ixy]\, dz$ along $y = x^3$, $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+9}\, dx$,

$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$, $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$, $\oint_c \frac{ze^z}{(z+2)^3}\, dz$, $|z| = 3$,

$\oint_c \frac{z^4}{(z+1)(z-i)^2}\, dz$, c is ellipse $9x^2 + 4y^2 = 36$, $\oint_c \frac{z+1}{(z^2+2z+4)}\, dz$, $|z+1+i| = 2$

ii- Find the Residues of the following functions:

$f(z) = \frac{z^2}{z^4-1}$, $f(z) = \frac{e^{2z}}{(z^2+9)^2(z^2+2z+2)^2}$, $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$

Question 2 **[20]**

Expand the function $f(z) = \frac{z}{(z^2-1)(z^2+4)}$ into a **Laurent series** in the power of z in the annulus

|z| > 2, then deduce the residues.

Question 3 **[20]**

i) **Find** the harmonic conjugate of $u = x^2 - y^2 + y$, then find f(z).

ii) **Find** the imaginary part of tanz

Question 4 **[60]**

Discuss briefly the following terms:

Laurent Series –Analytic function – Stretching and rotation transformations - Removable singularities - Essential singularities- Residues – Pole- Polar form of Cauchy Riemann equations (Derive) – Calculation of Residues – Linear and Bilinear transformations - Cauchy theorem (**state and prove**).

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Answer of question 1

- $\int_{1+i}^{2+8i} [x^2 + ixy]\, dz = \int_{1+i}^{2+8i} [x^2 + ixy][dx + idy] = \int_{1+i}^{2+8i} x^2dx - xydy] + i \int_{1+i}^{2+8i} xydx + x^2dy]$

Since the contour is $y = x^3$, therefore $dy = 3\, x^2\, dx$, hence

$$\int_{1+i}^{2+8i} [x^2 + ixy]\, dz \; = \int_1^2 [x^2 - 3x^6]\, dx + i \int_1^2 [4x^4]\, dx = \frac{7}{3} - \frac{384}{7} + \frac{124}{5} = -\frac{2911}{105}$$

- Since $\int_0^{\infty} \frac{\cos x}{x^2+9}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{x^2+9}\, dx$

Consider $\oint_c \frac{e^{iz}}{z^2+9} dz$, c consisting of the line along the x axis from –R to R and the semi circle above x axis, hence only z = i inside c such that $\operatorname*{Res}_{z=3i} f(z) = \lim_{z \to 3i} (z-3i) \frac{e^{iz}}{z^2+9} = \frac{e^{-3}}{6i}$, therefore

$$\oint_c \frac{e^{iz}}{z^2+9} dz = 2\pi i \operatorname*{Res}_{z=3i} = 2\pi i \frac{e^{-3}}{6i} = \frac{\pi}{3} e^{-3}, \text{ hence}$$

$$\int_{-R}^{R} \frac{e^{ix}}{x^2+9} dx = \frac{\pi}{3} e^{-3} = \int_{-R}^{R} \frac{\cos x}{x^2+9} dx + i \int_{-R}^{R} \frac{\sin x}{x^2+9} dx, \text{ thus}$$

$$2 \int_0^{R} \frac{\cos x}{x^2+9} dx = \frac{\pi}{3} e^{-3}, \text{ but } R \to \infty, \text{ therefore } \int_0^{\infty} \frac{\cos x}{x^2+9}\, dx = \frac{\pi}{6} e^{-3}$$

- $5 - 4\cos\theta \; = 5 - 4 \; (\frac{z + z^{-1}}{2}) = \frac{-2z^2 + 5z - 2}{z}$, therefore

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \frac{-1}{2i} \int_c \frac{(z^6+1)dz}{2z^2 - 5z + 2} = \frac{-1}{2i} \int_c \frac{(z^6+1)dz}{(2z-1)(z-2)}$$

$z = \frac{1}{2}, 2$ are inside the contour c , therefore the residue at $z = \frac{1}{2}, 2$ are $\frac{65}{192i}, \frac{-65}{6i}$ hence

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \pi(\frac{65}{96} - \frac{65}{3})$$

- Since z = -2 inside the contour, therefore $\oint_c \frac{ze^z}{(z+2)^3} dz = \pi i[ze^z + e^z]_{z=-2} = -\pi i e^{-2}$

- Let $f(z) = \frac{z^2}{(z^2 + a^2)(z^2 + b^2)}$, where $z = \pm ai, \pm bi$ are simple poles, but the contour is the upper half of complex plane from –R to R, therefore we have to get the residue at z = ai, bi

$$\operatorname{Res}_{z=ai} = \lim_{z \to ai} (z - ai) \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} = \frac{a^2}{2ai(-a^2 + b^2)} = \frac{a}{2i(a^2 - b^2)}$$. From symmetry ,

$\operatorname{Res}_{z=bi} = \frac{-b}{2i(a^2 - b^2)}$, thus $\int_{-R}^{R} \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} dz = 2\pi i(\operatorname{Res}_{z=ai} + \operatorname{Res}_{z=bi}) = \frac{\pi}{a+b}$ as $R \to \infty$,

therefore $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a+b}$

- Since the 2 poles are inside the contour, therefore $\oint_c \frac{z^4}{(z+1)(z-i)^2} dz = 2\pi i \sum \operatorname{Res}$

$\operatorname{Res}_{z=-1} = \lim_{z \to -1} \frac{z^4}{(z-i)^2} = \frac{1}{(1+i)^2} = -\frac{i}{2}$, $\operatorname{Res}_{z=i} = \lim_{z \to i} \frac{d}{dz}[\frac{z^4}{(z+1)}] = -\frac{4+3i}{2}$

Therefore $\oint_c \frac{z^4}{(z+1)(z-i)^2} dz = 4\pi[1-i]$

z = -1+$\sqrt{3}$ i is outside the contour, while z = -1+$\sqrt{3}$ i is inside the contour, therefore

$$\oint_c \frac{z+1}{(z^2 + 2z + 4)} dz = \oint_c \frac{[z+1]/[z - (-1 + \sqrt{3}i)]}{z + (1 + \sqrt{3}i)} dz = \pi i$$

Answer of question 2

$$\frac{z}{(z^2-1)(z^2+4)} = \frac{1}{5}\left[\frac{1}{(z^2-1)} - \frac{1}{(z^2+4)}\right] = \frac{1}{5z^2}\left[\frac{1}{(1-\frac{1}{z^2})} - \frac{1}{(1+\frac{4}{z^2})}\right]$$

$$= \frac{1}{5z^2}\left[(1+\frac{1}{z^2}+\frac{1}{z^4}+....) - (1-\frac{4}{z^2}+\frac{16}{z^4}-...)\right]$$

Answer of question 3

i) Since $u = x^2 - y^2 + y$ is harmonic, therefore $u_x = v_y = 2x$, therefore $v = 2xy + \phi(x)$ and hence $v_x = 2y + \phi`(x) = - u_y = 2y - 1 \Rightarrow \phi`(x) = -1 \Rightarrow \phi(x) = -x$.

Therefore $f(z) = x^2 - y^2 + y + i\,(2xy - x)$, thus $f(z) = z^2 - i\,z$

ii) $\tan z = \tan(x+iy) = \dfrac{\tan x + \tan(iy)}{1-\tan x\tan(iy)} = \dfrac{\tan x + i\tanh(y)}{1-i\tan x\tanh(y)}$

$= \dfrac{[\tan x + i\tanh(y)][1+i\tan x\tanh(y)]}{1+\tan^2 x\tanh^2(y)}$, therefore the imaginary part = $\dfrac{[\tan^2 x+1]\tanh(y)}{1+\tan^2 x\tanh^2(y)}$

Answer of question 4

Stretching : ($w = a\,z$)

By this transformation, figures in the z plane are stretched or (or contracted) in the direction z if $a > 1$ (or $0 < a < 1$). We contraction as a special case of stretching.

Rotation transformations: $w = e^{i\,\theta}\,z$

By this transformation, figures in the z plane are rotated through an angle θ. if $\theta > 0$, then rotation is counterclockwise, while if $\theta > 0$ the rotation is clockwise.

Laurent Series: If f(z) has a pole of order n at z = a , but is analytic at every other point inside and on a circle c with center at a , then $(z-a)^n f(z)$ is analytic at all points inside and on c and has Taylor series about z= a so that :

$f(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + ... + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + ...$ called Laurent Series for $f(z)$ and $a_0 + a_1(z-a) + a_2(z-a)^2 + ...$ is called analytic part, the remaining is called principle part, $z = a$ is called pole of order n.

Analytic function: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R.

Removable singularities: If a single–valued function f(z) is not defined at z = a , but $\lim_{z\to a} \frac{f(z)}{z-a}$ exist, z = a called removable singularity

Essential singularities: If f(z) is single–valued, then any singularity which is not a pole or removable singularity is called essential singularity.

Residues: Let f(z) be single-valued and analytic inside and on a circle c except at the point z = a chosen as center of c , then the Laurent series about z = a is given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$, where a_{-1} is the residue of f(z)

Polar form of Cauchy Riemann equations: In Polar form

Cauchy Riemann equations are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$, where $x = r\cos\theta$, $y = r\sin\theta$

Proof

Since $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$, $\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$ $\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}(\cos\theta) + \frac{\partial v}{\partial y}(\sin\theta)$, $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$, but $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, therefore $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Calculation of Residues:

1st case: (Simple pole)

The residue of the function f(z) at z =a which is simple pole (pole of order one) is given by

$$\operatorname*{Res}_{z=a} = a_{-1} = \lim_{z\to a}(z - a)f(z)$$

2^{nd} case: (pole of order n)

The residue of the function f(z) at z =a which is pole of order n is given by

$$\operatorname*{Res}_{z=a} = a_{-1} = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}(z - a)^{n} f(z)$$

Linear and Bilinear transformations:

Cauchy theorem: If a function f(z) is analytic inside and on a simple closed curve C and f\`(z) is continuous inside and on the curve C, then $\oint_c f(z)\,dz = 0$

Proof

$\oint_c f(z)\,dz = \oint_c u\,dx - v\,dy + i\oint_c v\,dx + u\,dy$ By Green \`s theorem , we can get

$\oint_c f(z)\,dz = \iint_R (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dx\,dy + \iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dx\,dy$, but f(z) is analytic inside and on a simple closed curve C, therefore f(z) satisfy Cauchy Riemann, i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, hence

$$\oint_c f(z)\,dz = 0$$