Benha University **Faculty of Engineering- Shoubra** **Mechanical Engineering Department** **1st Year Mechanical power** تخلفات		**Final Term Exam** **Date: 25th of May 2016** **Mathematics (2-B) - EMP 133** **Duration : 3 hours**

Answer all the following questions	• No. of questions: 2
The exam. Consists of one page	• Total Mark: 100 Marks

Question 1 [50]

a) Evaluate the following integrals $\int_0^\infty t\cos 3t\, e^{-5t}\, dt$, $\int_0^\infty \frac{t^2\,dt}{1+t^4}$, $\int_0^\infty x^n\, e^{-\sqrt{ax}}\, dx$

b) Solve the following differential equations using Laplace Transform:
i) $3y' + 4y = e^{2t}$, $y(0) = 1/3$, ii) $y'' + y = 2t$, $y(0) = 3$, $y'(0) = 1$

c) Solve the D.E. $y'' - xy' + y = 0$ using series solution about $x = x_0$

d) Find inverse Laplace for the following functions:

$$F(s) = \frac{e^{-2\pi s}}{(s+1)^2+4} + \frac{9s+4}{(s+3)^3}, \qquad G(s) = \frac{25}{s^3(s^2+4s+4)} + \frac{s}{s^2+4s+9}$$

e) Find Laplace transform for the following function:

$$f(t) = t\int_{u=0}^{t} \frac{e^{2u}-e^{-3u}}{u}du + [-e^{5t} + 2 + 3t^2]\,U(t-3)$$

Question 2 [50]

a) Suppose the probability that a college freshman will graduate is 0.6. Three sisters (triplets) enter college at the same time. What is the probability that at most 2 sisters will graduate?

b) In a factory we have four machines producing 1000, 1200, 1800, 2000 items per day with defects 1%, 5%, 5%, 1% respectively, find :

i) The probability of selecting a defective item.
ii) The probability that this defective item is produced by third machine.

c) Let the r.v. X be the distance in feet between bad records on a used computer tape. Suppose that a reasonable probability model for X is given by the p.d.f. $f(x) = \frac{1}{40}e^{-x/40}$, $0 < x < \infty$, find $P(X > x)$ and then the median, also find m.g.f. and then deduce mean, standard deviation.

d) A random variable X has the following distribution

X	-2	-1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	3k

Find $P(X < 2)$, $P(-2 < X < 2)$, C.d.f. , the mean of X and the variance.

e) Find the moment generating functions of the exponential distribution. Find the expected value, variance using M.G.F. and the first three moments about zero and about mean.

Board of Examiners **Dr. eng. Khaled El Naggar**

Model answer

Answer of question 1

a)

$$\int_0^\infty t\cos 3t\, e^{-5t}\,dt = L\{t\cos 3t\}_{s=5} = \left.\frac{s^2-9}{(s^2+9)^2}\right|_{s=5} = 4/289$$

$\int_0^\infty \frac{t^2\,dt}{1+t^4} = \frac{1}{4}\int_0^\infty \frac{y^{-1/4}\,dy}{1+y}$ (by putting $t^4=y$, hence $dt = \frac{1}{4}y^{-3/4}$), m-1=-1/4 & m +n = 1, thus

m = ¾, n = ¼, from which $\int_0^\infty \frac{t^2\,dt}{1+t^4} = \frac{1}{4}\int_0^\infty \frac{y^{-1/4}\,dy}{1+y} = \frac{1}{4}\beta(3/4,1/4) = \frac{\sqrt{2}}{4}\pi$

$\int_0^\infty x^n e^{-\sqrt{ax}}\,dx = \frac{2}{a^{n+1}}\int_0^\infty y^{2n+1}e^{-y}\,dy$ (by putting $\sqrt{ax}=y$, hence a dx = 2y dy), therefore

$\int_0^\infty x^n e^{-\sqrt{ax}}\,dx = \frac{2}{a^{n+1}}\Gamma(2n+2)$ b- By taking Laplace for Both equations , therefore

i) 3[sY(s) – y(0)] + 4 Y(s) = 1/(s-2), therefore $Y(s) = \frac{1}{(3s+4)(s-2)} + \frac{1}{(3s+4)}$, thus

$$Y(s) = \frac{1}{3[s^2-(2/3)s-(8/3)]} + \frac{1}{3[s+(4/3)]} = \frac{1}{3([s-(1/3)]^2-(25/9))} + \frac{1}{3[s+(4/3)]}, \text{ therefore}$$

$$y(t) = \frac{1}{5}e^{(1/3)t}\sinh(5/3)t + \frac{1}{3}e^{(-4/3)t}$$

ii) $(s^2+1)\,Y(s) - 3s - 1 = \frac{2}{s^2}$, therefore $Y(s) = \frac{2}{s^2(s^2+1)} + \frac{3s+1}{(s^2+1)}$, therefore

$$y(t) = \int_{u=0}^{t}(1-\cos u)\,du + 3\cos t + \sin t = 2[t - \sin t] + 3\cos t + \sin t$$

c) Let $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n,\ y'(x) = \sum_{n=1}^{\infty} n\,a_n(x-x_0)^{n-1},\ y''(x) = \sum_{n=2}^{\infty} n(n-1)\,a_n(x-x_0)^{n-2}$

Substitute in the above D.E., we get

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} - x\sum_{n=1}^{\infty} n\,a_n(x-x_0)^{n-1} + \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} -$$

$$\sum_{n=1}^{\infty} n\,a_n(x-x_0)^n - x_0\sum_{n=1}^{\infty} n\,a_n(x-x_0)^{n-1} + \sum_{n=0}^{\infty} a_n(x-x_0)^n = 0.$$

Put n = s+2 for 1st term, n = s for the 2nd term, n = s+1 for 3rd term and n = s for 4th term, we get:

$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}(x-x_0)^s - \sum_{s=1}^{\infty} s\, a_s(x-x_0)^s$$

$$x_0\sum_{s=0}^{\infty}(s+1)a_{s+1}(x-x_0)^s + \sum_{s=0}^{\infty}a_s(x-x_0)^s = 0,$$ thus $2a_2 - x_0\ a_1 + a_0 +$

$$\sum_{s=1}^{\infty}((s+2)(s+1)a_{s+2} - x_0(s+1)a_{s+1} - (s-1)a_s)(x-x_0)^s = 0.$$

By comparing of coefficients, we get:

$2a_2 - x_0\ a_1 + a_0 = 0$, from which $a_2 = \dfrac{x_0\ a_1 - a_0}{2}$ and by comparing coefficients of $(x-x_0)^s$, s = 1, 2, 3, ….,we get

$$a_{s+2} = \frac{x_0(s+1)a_{s+1} + (s-1)a_s}{(s+2)(s+1)}$$

Therefore

$$a_3 = \frac{x_0\ a_2}{3} = \frac{x_0(x_0 a_1 - a_0)}{6},\ a_4 = \frac{3x_0 a_3 + a_2}{12} = \frac{(x_0^2+1)(x_0\ a_1 - a_0)}{24},$$

The solution will be in the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = A\left(1 - \frac{1}{2}(x-x_0)^2 - \frac{x_0}{6}(x-x_0)^3 - \frac{(x_0^2+1)}{24}(x-x_0)^4 + \ldots\right) +$$
$$B\left((x-x_0) + \frac{x_0}{2}(x-x_0)^2 + \frac{x_0^2}{6}(x-x_0)^3 + \frac{(x_0^2+1)\,x_0}{24}(x-x_0)^4 + \ldots\right),$$ where $a_0 = A$, $a_1 = B$,

d) $L^{-1}\{\dfrac{e^{-2\pi s}}{(s+1)^2+4}\} = (½) \sin 2(t-2\pi) e^{-(t-2\pi)}\ u(t-2\pi)$ and Since $\dfrac{9s+4}{(s+3)^3} = \dfrac{9(s+3)-23}{(s+3)^3}$,

therefore $L^{-1}\{\dfrac{9s+4}{(s+3)^3}) = 9t\,e^{-3t}-(23/2)$

Since $L^{-1}\{\dfrac{25}{s(s+2)^2}\} = 25\int_{u=0}^{t} u\, e^{-2u}\, du = \dfrac{-25}{4}[(2t+1)e^{-2t} - 1]$, thus $L^{-1}\{\dfrac{1}{s^2(s+2)^2}\}$

$= (-25/4) \int_{u=0}^{t} [(2u+1)e^{-2u} - 1]\, du = (25/4)[(t+1)\, e^{-2t} + t - 1]$, therefore $L^{-1}\{\frac{1}{s^3(s+2)^2}\} =$

$(25/4) \int_{u=0}^{t} [(u+1)e^{-2u} + u - 1]\, du$

Since $\frac{s}{s^2+4s+9} = \frac{s+2-2}{(s+2)^2-4+9}$, therefore $L^{-1}\{\frac{s}{s^2+4s+9}\} = e^{-2t}\cos\sqrt{5}\, t - \frac{2}{\sqrt{5}} e^{-2t}\sin\sqrt{5}\, t$

$= \frac{1}{9}[t + \frac{e^{-9t}}{9} - \frac{1}{9}] + e^{-2t}[\cos\sqrt{5}\, t - \frac{2}{\sqrt{5}}\sin\sqrt{5}\, t]$

e) $L\{\frac{e^{2t} - e^{-3t}}{t}\} = \int_{s}^{\infty} [\frac{1}{s-2} - \frac{1}{s+3}] ds = \mathrm{Ln}\left|\frac{s+3}{s-2}\right|$, thus $L\{\int_{u=0}^{t} \frac{e^{2u} - e^{-3u}}{u} du\} = \frac{1}{s}\mathrm{Ln}\left|\frac{s+3}{s-2}\right|$

Therefore $L\{t \int_{u=0}^{t} \frac{e^{2u} - e^{-3u}}{u} du\} = -\frac{d}{ds}\{\frac{1}{s}\mathrm{Ln}\left|\frac{s+3}{s-2}\right|\}$

Since $-e^{5t} + 2 + 3t^2 = -e^{15} e^{5(t-3)} + 29 + 3(t-3)^2 + 18(t-3)$, therefore $L\{[-e^{5t} + 2 + 3t^2]\, U(t-3)\}$

$= \frac{-e^{15}}{s-5} + \frac{29}{s} + \frac{6}{s^3} + \frac{18}{s^2}$

Answer of question 2

a) $P = 0.6$, $n = 3$, $P(X \le 2) = \sum_{x=0}^{2} {}^{3}c_x (0.6)^x (0.4)^{3-x} = 1 - P(X=3) = 1 - {}^{3}c_3 (0.6)^3 (0.4)^0 = 0.784$

b) Let the defective event is F and the probability of machines A,B,C,D are 1/6, 1/5, 3/10, 1/3 respectively, also P(F/A) = 0.01, P(F/B) = 0.05, P(F/C) = 0.05, P(F/D) = 0.01. Therefore

i- P(F) = P(F/A)P(A) + P(F/B)P(B) + P(F/C)P(C) + P(F/D)P(D) = 0.01(1/6) + 0.05(1/5) + 0.05(3/10) + 0.01(1/3) = 0.03

ii- $P(C/F) = \frac{P(F/C)P(C)}{P(F)} = \frac{0.05(3/10)}{0.03} = 0.5$

c) $P(X > x) = 1 - P(X<x) = 1 - \int_{0}^{x} \frac{1}{40} e^{-x/40} dx = e^{-x/40}$. To get the median a such that $P(X< a) = 0.5$, therefore $\int_{0}^{a} \frac{1}{40} e^{-x/40} dx = 0.5$, thus $1 - e^{-a/40} = 0.5 \Rightarrow a = -40 \ln(0.5) = 27.726$, and

m.g.f. $= \int_{0}^{\infty} e^{tx} (\frac{1}{40} e^{-x/40}) dx = \int_{0}^{\infty} \frac{1}{40} e^{\frac{-(1-40t)x}{40}} dx = \frac{1}{1-40t} = \phi(t)$, therefore $\mu_1' = \phi'(0) =$

$\left.\frac{40}{(1-40t)^2}\right|_{t=0}$ = 40 = E(X) and $\mu_2' = \phi''(0) = \left.\frac{3200}{(1-40t)^3}\right|_{t=0}$ = 3200 = $E(X^2)$, hence var(X) = $E(X^2) - [E(X)]^2$ = 3200 – 1600 = 1600, so standard deviation = 40

d) Since $\sum_{i=-2}^{3} P(x_i) = 1$, therefore 0.1 + k + 0.2 + 2k + 0.3 + 3k = 1, thus k = 1/15

P (X < 2) = P(x = -2) + P(x = -1) + P(x=0) + P(x=1) = 0.3 + 3k = 0.5

P (-2 < X < 2) = P(x = -1) + P(x=0) + P(x=1) = 3k + 0.2 = 0.4

cumulative distribution of X

X	-2	-1	0	1	2	3
F(x)	0.1	1/6	11/30	0.5	0.8	1

E(X) = -2(0.1) -1(1/15) + 0(0.2) + 1(2/15) + 2(0.3) + 3(1/5) = 16/15, $E(X^2)$ = 4(0.1) +1(1/15) + 0(0.2) + 1(2/15) + 4(0.3) + 9(1/5) = 2.4, thus Var(X) = $E(X^2) - [E(X)]^2$ = 2.4 – $(16/15)^2$ = 1.2622.

e) $E(e^{tx}) = \int_0^\infty e^{tx}(\lambda e^{-\lambda x})\,dx = \lambda\int_0^\infty e^{-(\lambda-t)x}\,dx = \frac{\lambda}{\lambda-t}$

$E(X) = \frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right) = \frac{\lambda}{(\lambda-t)^2}$, at t =0 E(X) = $1/\lambda$

$E(X^2) = \frac{d^2}{dt^2}\left(\frac{\lambda}{\lambda-t}\right) = \frac{2\lambda}{(\lambda-t)^3}$, at t =0 $E(X^2) = 2/\lambda^2$, Var(X) = $1/\lambda^2$

Moments about zero: $\mu'_0 = 1$, $\mu'_1 = E(X) = 1/\lambda$, $\mu'_2 = E(X^2) = 2/\lambda^2$

Moments about zero: $\mu_0 = 1$, $\mu_1 = 0$, $\mu_2 = \text{var}(X) = 1/\lambda^2$