Benha University
Faculty of Engineering- Shoubra
Electrical Engineering Department
1st Year electrical power تخلفات

Final Term Exam
Date:23rd of May 2016
Mathematics 2A EMP 173
Duration : 3 hours

- Answer all the following question
- Illustrate your answers with sketches when necessary.
- The exam. Consists of one page
- No. of questions:6
- Total Mark: 100 Marks

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1) Evaluate the following integrals **[20 Marks]**

a) $\int_0^\infty \frac{t^2\,dt}{1+t^4}$, b) $\int_a^b (x-a)^m (b-x)^n\,dx$, c) $\int_0^{\pi/2} (\tan x)^n\,dx$, d) $\int_0^\infty t\sin 2t\cos 3t\, e^{-4t}\,dt$

2) Find inverse Laplace for the following functions **[20 Marks]**

$$F(s) = \frac{s\,e^{-2s}}{s^2+6s+20} + \frac{1}{(3s+4)^3}, \qquad G(s) = \frac{1}{s^3(s^2+9)} + \frac{3s+2}{s^2+6s+5}$$

3) Find Series solution for the following D.E. about x = 0 **[20 Marks]**

$$(1+x^2)\,y'' + y = x$$

4) Solve the following differential equations using L.T. **[10 Marks]**

i) $3y' + 4y = e^{2t}$, $y(0) = 1/3$, ii) $y'' + y = 2t$, $y(0) = 3$, $y'(0) = 1$

5) Solve the following P.D.E'S **[10 Marks]**

a) $3u_x + 4u_y - 5u = 10(y + x)$

b) $9u_{xx} = u_{tt}$, B.C. $u(0,t) = u(2,t) = 0$, I.C. $u(x,0) = x-1$, $u_t(x,0) = 4x$

6) Find Laplace transform for the following function **[20 Marks]**

i) $f(t) = t\sin 2t\cosh 3t + e^{-3t}\int_{u=0}^{t} e^{-2u}\sin u\,du$, ii) $g(t) = \begin{cases} 6 & 0 < t \le 2 \\ 3t & 2 < t \le 3 \\ 9 & t > 3 \end{cases} + \frac{\cos 2t - \cos 3t}{t}$

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Model answer

Answer of question 1

a) $\int_0^\infty \frac{t^2 dt}{1+t^4} = \frac{1}{4}\int_0^\infty \frac{y^{-1/4} dy}{1+y}$ (by putting $t^4 = y$, hence dt = $\frac{1}{4}$ $y^{-3/4}$), therefore m-1 = -1/4 and m + n = 1, thus m = ¾, n = ¼, from which $\int_0^\infty \frac{t^2 dt}{1+t^4} = \frac{1}{4}\int_0^\infty \frac{y^{-1/4} dy}{1+y} =$

$\frac{1}{4}$ β(3/4,1/4)= $\frac{\sqrt{2}}{4}\pi$.

b) Put x-a = y $\Rightarrow$ dx = dy, therefore

$$\int_a^b (x-a)^m (b-x)^n dx = \int_0^{b-a} y^m [(b-a)-y]^n dy = (b-a)^n \int_0^{b-a} y^m [1-\frac{y}{b-a}]^n dy .$$

put z = $\frac{y}{b-a}$ $\Rightarrow$ (b-a) dz = dy, thus

$$\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1}\int_0^1 z^m [1-z]^n dz = (b-a)^{m+n+1}\beta(m+1, n+1)$$

c) $\int_0^{\pi/2} (\tan x)^n dx = \int_0^{\pi/2} (\sin\theta)^n (\cos\theta)^{-n} d\theta = \frac{1}{2}\beta(p,q)$, such that 2p-1 = n and 2q-1 = -n,

therefore p = (n+1)/2, q = (1-n)/2, thus $\int_0^{\pi/2} (\tan x)^n dx = \frac{1}{2}\beta((n+1)/2, (1-n)/2)$

$$= \frac{\pi}{2\sin(\frac{n+1}{2})} = \frac{2^{1/4}}{2}\beta(5/8, 1/2)$$

d) $\int_0^\infty t \sin 2t \cos 3t\, e^{-4t}\, dt = \frac{1}{2}\int_0^\infty t[-\sin t + \sin 5t]e^{-4t}\, dt$

Since L{t sint}= $\frac{2s}{(s^2+1)^2}$, L{t sin5t}= $\frac{10s}{(s^2+25)^2}$, therefore

$$\int_0^\infty t \sin 2t \cos 3t\, e^{-4t}\, dt = \frac{1}{2}[-\frac{2s}{(s^2+1)^2} + \frac{10s}{(s^2+25)^2}]_{s=4} = -\frac{4}{(17)^2} + \frac{20}{(41)^2}$$

Answer of question 2

a) $F(s) = \frac{s\, e^{-2s}}{s^2+6s+20} + \frac{1}{(3s+4)^3} = \frac{(s+3-3)\, e^{-2s}}{(s+3)^2+11} + \frac{1}{27(s+4/3)^3}$, therefore

$$f(t) = e^{-3(t-2)}[\cos\sqrt{11}(t-2) - \frac{3}{\sqrt{11}}\sin\sqrt{11}(t-2)]U(t-2) + \frac{1}{54}e^{-(4/3)t}t^2$$

$$G(s) = \frac{1}{s^3(s^2+9)} + \frac{3s+2}{s^2+6s+5} = \frac{1}{s^3(s^2+9)} + \frac{3(s+3-3)+2}{(s+3)^2-4}$$, therefore

$$g(t) = \frac{1}{9}\int_{u=0}^{t}(u - \frac{\sin 3u}{3})\ du + 3e^{-3t}\cosh 2t - \frac{7}{2}e^{-3t}\sinh 2t = \frac{1}{9}[\frac{t^2}{2} + \frac{\cos 3t}{9}] +$$

$$3e^{-3t}\cosh 2t - \frac{7}{2}e^{-3t}\sinh 2t$$

Answer of question 3

Since p(x) and q(x) are analytic for all x, therefore $x = x_0$ is regular point and substitute (4) in the above D.E. , we will get

$$(1+x_0^2)\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^n + 2x_0\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-1}$$

$$+ \sum_{n=0}^{\infty} a_n(x-x_0)^n = x$$

Put n = s+2 for the 1st term, n = s for 2nd & 4th terms, n = s+1 for 3rd term, we get

$$(1+x_0^2)\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}(x-x_0)^s + \sum_{s=2}^{\infty} s(s-1)a_s(x-x_0)^s$$

$$+ 2x_0\sum_{s=0}^{\infty} s(s+1)a_{s+1}(x-x_0)^s + \sum_{s=0}^{\infty} a_s(x-x_0)^s = x$$, therefore $a_0 + 2(1+x_0^2)\ a_2 +$

$[a_1 + 4x_0 a_2 + 6(1+x_0^2)a_3]\ (x-x_0)\ +$

$$\sum_{s=2}^{\infty}((1+x_0^2)(s+2)(s+1)a_{s+2} + 2x_0 s(s+1)a_{s+1} + (s^2-s+1)a_s)(x-x_0)^s = x.$$

By comparing of coefficients we get

$$a_2 = \frac{x_0 - a_0}{2(1+x_0^2)},\quad a_3 = \frac{2(1-(1+x_0^2)a_1 + x_0 a_0)}{12(1+x_0^2)^2}$$

$$(1+x_0^2)(s+2)(s+1)a_{s+2} + 2x_0 s(s+1)a_{s+1} + (s^2-s+1)a_s = 0, s=2,3,..$$

At s =2, $12(1+x_0^2)a_4 + 12x_0 a_3 + 3a_2 = 0$, therefore

$$a_4 = \frac{x_0^2 - 3}{24(1+x_0^2)^3}\, a_0 - \frac{x_0}{6(1+x_0^2)^2}\, a_1 + \frac{7x_0+3x_0^3}{2(1+x_0^2)^2}$$

The solution will be in the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Let $a_0 = A$, $a_1 = B$, therefore

$$y(x)=A\ [\ 1 + \frac{-1}{2(1+x_0^2)}(x - x_0)^2 + \frac{x_0}{6(1+x_0^2)^2}(x-x_0)^3 + \frac{x_0^2 - 3}{24(1+x_0^2)^3}(x - x_0)^4 + ...] +$$

$$B((x - x_0) - \frac{1}{6(1+x_0^2)}(x-x_0)^3 - \frac{x_0}{6(1+x_0^2)^2}(x - x_0)^4 + ...) + \frac{x_0}{2(1+x_0^2)}(x-x_0)^2$$

$$+\frac{1}{6(1+x_0^2)^2}(x-x_0)^3 + \frac{7x_0+3x_0^3}{2(1+x_0^2)^2} + ...,$$ put $x_0 = 0$

Answer of question 4

By taking Laplace for Both equations , therefore

i) $3[\ sY(s) - y(0)] + 4\ Y(s) = 1/(s-2)$, therefore $Y(s) = \frac{1}{(3s+4)(s-2)} + \frac{1}{(3s+4)}$, thus

$$Y(s) = \frac{1}{3[s^2 - (2/3)s - (8/3)]} + \frac{1}{3[s+(4/3)]} = \frac{1}{3\,([s-(1/3)]^2 - (25/9))} + \frac{1}{3[s+(4/3)]},$$

therefore $y(t) = \frac{1}{5}\ e^{(1/3)t}\ \sinh(5/3)t + \frac{1}{3}\ e^{(-4/3)t}$

ii) $(s^2 + 1)\ Y(s) - 3s - 1 = \frac{2}{s^2}$, therefore $Y(s) = \frac{2}{s^2(s^2+1)} + \frac{3s+1}{(s^2+1)}$, therefore

$$y(t) = \int_{u=0}^{t} (1 - \cos u)\ du + 3\cos t + \sin t = 2[t - \sin t] + 3\cos t + \sin t$$

Answer of question 5

a) The solution will be in the form: $w_\alpha + k\,w = g(\alpha, \beta)$ such that:

$\alpha = x\cos\theta + y\sin\theta$, $\beta = -x\sin\theta + y\cos\theta \Rightarrow x = \alpha\ \cos\theta - \beta\ \sin\theta$, $y = \alpha\ \sin\theta + \beta$ $\cos\theta$,

$u_x = u_\alpha \ \alpha_x + u_\beta \ \beta_x = u_\alpha \cos\theta - u_\beta \sin\theta$,

$u_y = u_\alpha \ \alpha_y + u_\beta \ \beta_y = u_\alpha \sin\theta + u_\beta \cos\theta$

Substitute in the above P.D.E. so that:

$3(w_\alpha \cos\theta - w_\beta \sin\theta) + 4(w_\alpha \sin\theta + w_\beta \cos\theta) - 5w = 10[\alpha(\cos\theta + \sin\theta) + \beta(\cos\theta - \sin\theta)]$

Coefficient of $w_\beta = 0$, therefore $-3\sin\theta + 4\cos\theta = 0$, thus $\tan\theta = 4/3$, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$, therefore $5w_\alpha - 5w = 14\alpha - 2\beta \Rightarrow w_\alpha - w = \frac{2}{5}[7\alpha - \beta]$

which is linear D.E. such that the solution will be in the form:

$$w(\alpha,\beta)e^{-\alpha} = \int \frac{2}{5}[7\alpha - \beta]e^{-\alpha} d\alpha + \phi(\beta)$$

b) $U(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{2})x[A_n \cos(\frac{3n\pi}{2})t + B_n \sin(\frac{3n\pi}{2})t]$

But $U(x,0) = x - 1 = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{2})x$, which is Fourier sine series such that

$$A_n = \int_0^2 (x-1)\sin(\frac{n\pi}{2})x \, dx$$

Since $U_t(x,t) = \sum_{n=1}^{\infty} (\frac{3n\pi}{2})\sin(\frac{n\pi}{2})x[-A_n \sin(\frac{3n\pi}{2})t + B_n \cos(\frac{3n\pi}{2})t]$

And $U_t(x,0) = 4x$, therefore $\sum_{n=1}^{\infty} B_n(\frac{3n\pi}{2})\sin(\frac{n\pi}{2})x = 4x$, which is Fourier sine series such that $B_n(\frac{3n\pi}{2}) = \int_0^L 4x \sin(\frac{n\pi}{2})x \, dx$, therefore

$$B_n = \frac{2}{3n\pi}\int_0^L 4x \sin(\frac{n\pi}{2})x \, dx$$

Answer of question 6

$$L\{ t\sin 2t\cosh 3t\} = L\{t\sin 2t(e^{3t} + e^{-3t})/2\} = \frac{2(s-3)}{[(s-3)^2+4]^2} + \frac{2(s+3)}{[(s+3)^2+4]^2}$$

$$L\{e^{-3t}\int_{u=0}^{t} e^{-2u}\sin u\,du\} = \frac{1}{[(s+5)^2+1][s+3]}$$

Since $g(t) = \begin{cases} 6 & 0 < t \le 2 \\ 3t & 2 < t \le 3 \\ 9 & t > 3 \end{cases} = 6u(t) + 3(t-2)\,U(t-2) - 3(t-3)\,U(t-3)$, therefore

$$G(s) = 6/s + (3/s^2)e^{-2s} - (3/s^2)e^{-3s}$$

$$L\{\frac{\cos 2t - \cos 3t}{t}\} = \int_s^{\infty}(\frac{s}{s^2+4} - \frac{s}{s^2+9})ds = \frac{1}{2}\ln[\frac{s^2+9}{s^2+4}]$$